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On the Evolution of the Large-Scale Structures in the Far Field of the Plane Jet

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Abstract

Gordeyev and Thomas[6] recently found that measurements of the large-scale structures, extracted using Lumley's POD, were self similar in the intermediate and far field of the plane jet as predicted by Ewing[1]. It is demonstrated here that the equations for the two-point velocity correlation tensor have self-similar solutions in the plane jet so that the orthogonal modes extracted using the POD should be self-similar as Gordeyev and Thomas found. It is also shown that the scaled two-point velocity correlation should be homogeneous in the mean flow direction if the coordinate is rescaled using a logarithmic function. Gordeyev and Thomas did not measured the correlation in the mean flow direction so this prediction could not be verified.

Introduction

There is currently considerable interest in understanding the dynamics of the large-scale structures in turbulent flows and developing simple models for the development of these structures. One significant challenge in developing low-dimensional models for turbulent shear flows is including the development of the flow in the mean flow direction. Currently, this is ignored and it is assumed the flow is homogeneous in the streamwise direction [11]. This simplifies the models but it is not clear whether the models include all of the important physics.

Spalart [9] incorporated the non-homogeneity in the mean flow direction in Direct Numerical Simulations of turbulent boundary layers by recognizing the profiles of the mean velocity and turbulent stresses evolve in a self-similar manner. This approach was developed using the single-point similarity theory so it only accounted for the growth of the boundary layer in the non-homogeneous shear direction. It is known, however, that the characteristic length scale of the large-scale structures grow in the streamwise direction at the same rate as the non-homogeneous direction. The characteristic time scale of these motions also change affecting the pressure feedback mechanisms in free-shear flows. Techniques must be developed to model these effects in order to determine their role in the development of the large-scale structures in turbulent shear flows.

One technique is to extend the similarity analysis of the governing equations to measures of the large-scale structures. Physically, similarity of the single-point moments implies the turbulent flow is evolving in an equilibrium manner. This can only be true, however, if the underlying turbulent structures are evolving in an equilibrium manner in which case the governing equations for more complex measures of the large-scale structures should also have self-similar solutions.

Ewing *et al.* [1, 2, 3] showed that the governing equations for the more general two-point velocity correlation tensor have self-similar solutions in several flows including the plane and axiysmmetric jets and the temporally plane wake. Ewing *et al.* [3] also showed the predictions of the theory were in good agreement with data from two direct numerical simulations of the temporally evolving wake. Recently, Gordeyev and Thomas[6] performed measurements of the two-point velocity correlations in the intermediate and far field of a high-Reynolds-number plane jet and used Lumely's Proper Orthogonal Decomposition[8] to extract modes to represent the large-scale structures. They found the POD modes and eigenvalues were self-similar as predicted by Ewing [1]. The analysis of the governing equations for the plane jet and the implications of the result on the representation of the large-scale motions are discussed here.

Self-Similarity Solution for the Two-point Correlations

The analysis of the plane jet differs from both the axisymmetric jet and the plane wake because the local Reynolds number of the flow changes as the flow evolves downstream. In particular, George [4] showed that the governing equations for the mean momentum and turbulent Reynolds stresses in the far field of a plane jet have self-similar when δ the jet-half width is given by $\delta \sim x_1$ and the mean velocity scale U_s is given by $U_s \sim x^{-1/2}$. Thus, the Reynolds of the flow given by

$$Re = \frac{U_c \delta}{\nu} \sim x_1^{1/2} \tag{1}$$

increases as x_1 the distance downstream of the jet's virtual origin increases. This causes the ratio of the size of the energy containing eddies and dissipation scales to change as the flow evolves downstream so a single set of self-similar solutions can not describe the evolution of both the large- and small-scale motions in the plane jet. Instead, following the approach outlined by Kolomogorov [7], the motions are divided into large-scale energy containing eddies and small-scale motions that dissipate the energy. The objective here is to determine if governing equations for measures of both these motions have self-similar solutions that are compatible with each other.

Large-Scale Motions

The evolution of the large-scale motions can be examined by considering the two-point velocity correlation for large separation distances. The first-order governing equations for these correlations in the far field of the plane jet is given by

$$\begin{aligned} U_1 \frac{\partial \overline{u_i u'_j}}{\partial x_1} + U'_1 \frac{\partial \overline{u_i u'_j}}{\partial x'_1} + U_2 \frac{\partial \overline{u_i u'_j}}{\partial x_2} + U'_2 \frac{\partial \overline{u_i u'_j}}{\partial x'_2} \\ = -\frac{1}{\rho}\delta_{i1}\frac{\partial \overline{p u'_j}}{\partial x_1} - \frac{1}{\rho}\delta_{i2}\frac{\partial \overline{p u'_j}}{\partial x_2} - \frac{1}{\rho}\delta_{i3}\frac{\partial \overline{p u'_j}}{\partial r_3} \\ -\frac{1}{\rho}\delta_{j1}\frac{\partial \overline{p' u_i}}{\partial x'_1} - \frac{1}{\rho}\delta_{j2}\frac{\partial \overline{p' u_i}}{\partial x'_2} + \frac{1}{\rho}\delta_{j3}\frac{\partial \overline{p' u_i}}{\partial r'_3} \end{aligned}$$

$$-\frac{\partial \overline{u_1 u_i u_j'}}{\partial x_1} - \frac{\partial \overline{u_2 u_i u_j'}}{\partial x_2} - \frac{\partial \overline{u_1' u_i u_j'}}{\partial x_1'} - \frac{\partial \overline{u_2' u_i u_j'}}{\partial x_2'}$$

$$-\frac{\partial \overline{(u_3 - u_3') u_i u_j'}}{\partial r_3} - \overline{u_j' u_k}\frac{\partial U_i}{\partial x_k} - \overline{u_i u_k'}\frac{\partial U_j'}{\partial x_k'}, \qquad (2)$$

where x_1 is the coordinate in the mean flow direction, x_2 is the coordinate in the cross-stream direction, x_3 is the coordinate in the homogeneous direction, and $r_3 = x_3 - x_3'$ is the separation distance in the x_3 direction. Here, U_i, u_i, U_j', and u_j' are the mean and fluctuating velocities evaluated at two independent points, x_i and x_i', in the jet

The mean continuity equation and the first order mean momentum equations given by

$$\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2} = 0 \qquad (3)$$

$$U_1 \frac{\partial U_1}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2} = -\frac{\partial \overline{u_1 u_2}}{\partial x_2}, \qquad (4)$$

and

$$\int U_1^2 dx_2 = M_o. \qquad (5)$$

must also considered in the analysis in order to determine the constraints on the scales for the mean velocity profile. Here, M_o is the momentum flux of the jet per unit width in the homogeneous x_3 direction. The similarity solution for the Reynolds shear stress $\overline{u_1 u_2}$ in these equations must be consistent with the solutions for the two-point velocity correlations $\overline{u_1 u_2'}$.

It is hypothesized that the equations for the two-point velocity correlation have self-similar solutions that are the product of a scale that depends on the position of the two points downstream of the jet exit and a similarity function that depends only on the separation distance between the points. Clearly, the similarity function can not depend on the separation between the two points in physical variables $x_1 - x_1'$ because the length scale of the motions in the streamwise direction are growing as the flow evolves downstream. Thus, the motions that should be self-similar have different characteristic length scales $x_1 - x_1'$ at different downstream positions.

Although it is not shown rigorously here, this decomposition can be accomplished by normalizing the separation distances in the streamwise direction, $x_1 - x_1'$, by the local length scale of the flow; i.e.,

$$d\xi = \frac{dx_1}{\delta(x_1)} \sim \frac{dx_1}{x_1}. \qquad (6)$$

Thus, the new coordinate for the streamwise direction is given by $\xi = ln(x_1/x_1^0)$. The similarity scales are then written as a function of the average downstream location of the two points in this transformed system; i.e.,

$$\bar{\xi} = \frac{1}{2}(\xi + \xi'). \qquad (7)$$

It is straightforward to show that the velocity and length scale at the downstream distance $\bar{\xi}$ can be written as

$$U_s(e^{\bar{\xi}}) = [U_s(x_1 U_s(x_1')]^{1/2} \qquad (8)$$

and

$$\delta(e^{\bar{\xi}}) = [\delta(x_1)\delta(x_1')]^{1/2}. \qquad (9)$$

Thus, the similarity scales at $\bar{\xi}$ are related to the length and velocity scales at the two points. The self-similar functions are then a function of the separation distance between the points in this coordinate system given by

$$\upsilon = \xi - \xi'. \qquad (10)$$

Thus, it is proposed that the governing equations for the two-point correlations have self-similar solutions of the form

$$U_1 = U_s(x_1) f(\eta) \qquad (11)$$

$$\overline{u_i u_j'} = Q^{i,j}(\xi) q_{i,j}(\upsilon, \eta, \eta', \zeta, *), \qquad (12)$$

$$\overline{u_k u_i u_j'} = T_1^{ki,j}(\xi) t_{ki,j}^1(\upsilon, \eta, \eta', \zeta, *), \qquad (13)$$

$$\overline{u_k' u_i u_j'} = T_2^{i,kj}(\xi) t_{i,kj}^2(\upsilon, \eta, \eta', \zeta, *), \qquad (14)$$

$$\overline{p u_j'} = \Pi_1^j(\xi) \pi_j^1(\upsilon, \eta, \eta', \zeta, *), \qquad (15)$$

$$\overline{p' u_i} = \Pi_2^i(\xi) \pi_i^2(\upsilon, \eta, \eta', \zeta, *). \qquad (16)$$

where η, η', and ζ are similarity variables given by

$$\eta = x_2/\delta(e^{\bar{\xi}}),\ \eta' = x_2'/\delta(e^{\bar{\xi}}),\ \text{and}\ \zeta = r_3/\delta_3(e^{\bar{\xi}}). \qquad (17)$$

Here, $*$ indicates any dependence on the jet's initial condition.

The governing equations have self-similar solutions of the proposed form if all of the terms in the equation have the same $\bar{\xi}$ dependence. In this case all the terms in the equation decay at the same rate indicating that the different energy transfer processes in the flow are evolving in an equilibrium manner. It is straightforward to show this condition is satisfied if

$$\delta_3(x_1) \propto \delta(x_1) \propto x_1, \qquad (18)$$

$$Q^{i,j}(\xi) \propto U_s(e^{\bar{\xi}})^2 = U_s(x_1) U_s(x_1'), \qquad (19)$$

$$T_1^{ki,j}(\xi) \propto U_s(e^{\bar{\xi}})^3, \qquad (20)$$

and similar constraints are satisfied for the other moments.

Physically, the first constraint implies that the characteristic length scales of the motions in all directions grow at the same rate when the structures are evolving in equilibrium manner. The growth of the structures in the lateral direction should be considered when designing either physical or numerical experiments to ensure the development of the structures are not affected by finite boundaries, including when boundary condition recycling is used in numerical simulations where the exit flow is rescaled and used for the entrance boundary conditions.

Pressure Field

The pressure at any point in an incompressible flow is a measure of events at all other locations so that the two-point pressure velocity correlation is a highly non-local term. The two-point pressure velocity correlation at any point can be written as[7]

$$\frac{\overline{p u_j'}}{\rho} = -\frac{1}{4\pi}\int 2\left[\frac{\partial U_k''}{\partial x_l''}\frac{\partial \overline{u_l'' u_j'}}{\partial x_k''}\right]\frac{dx_1'' dx_2'' dx_3''}{|x_\alpha'' - x_\alpha|}$$

$$-\frac{1}{4\pi}\int \frac{\partial^2 \overline{u_k'' u_l'' u_j'}}{\partial x_k'' \partial x_l''}\frac{dx_1'' dx_2'' dx_3''}{|x_\alpha'' - x_\alpha|}. \qquad (21)$$

With some effort it can be shown this equation can be used to develop an equation for the similarity solution $\pi_j^1(\upsilon, \eta, \eta', \zeta, *)$ that only depends on similarity variables. This may not seem surprising since the integral equation for the pressure-velocity correlation can be developed directly from the equation for the two-point velocity correlation. This results emphasizes, however, that the energy transfer processes at each location in the flow are in equilibrium with the processes at all other locations in the flow. It should be noted that the integral equation was developed assuming that the flow was evolving in an infinite environment. This assumption is appropriate because the self-similar solutions outlined here represent at most a asymptotic state for a plane jet in an infinite environment.

Small-Scale Motions

Following the approach outlined by Kolomogorov, the evolution of the small-scale motions is examined using the second-order structure function. It can be shown that the first order equation for this structure function in the plane jet is given by

$$+4\overline{\frac{p}{\rho}\frac{\partial u_\alpha}{\partial \bar{x}_\alpha}} - 2\epsilon_{\alpha\alpha} = -\frac{\partial\overline{\delta u_k(\delta u_\alpha)^2}}{\partial x_k}$$

$$-\frac{2}{\rho}\frac{\partial}{\partial r_\alpha}\left[\overline{pu'_\alpha - p'u_\alpha}\right] - 2\nu\frac{\partial^2}{\partial \tilde{x}_k^2}\overline{(\delta u_\alpha)^2}, \qquad (22)$$

where $\delta u_\alpha = (u_\alpha - u'_\alpha)$, r_α is the separation between the two points, and $\bar{x}_\alpha$ is the central location of the two points. The production terms in this equation are neglected since they should be small relative to the dissipation terms. Physically, the dissipation and pressure strain terms represent the energy transfer from the large-scale motions to the small-scale motions. It follows from the analysis of the large-scale motions and the single-point equations that

$$\epsilon_{\alpha\alpha} \sim \overline{\frac{p}{\rho}\frac{\partial u_\alpha}{\partial x_\alpha}} \sim \frac{U_s^3}{\delta}\frac{d\delta}{dx}, \qquad (23)$$

which is proportional to u^3/δ in this flow since the growth rate is constant.

It is hypothesized that the equation for the structure functions have self-similar solutions given by

$$\overline{(\delta u_\alpha)^2} = B^{\alpha\alpha}(\bar{x}_1) b_{\alpha\alpha}(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \eta, *), \qquad (24)$$

$$\overline{\delta u_k(\delta u_\alpha)^2} = T^{\alpha\alpha k}(\bar{x}_1) t_{\alpha\alpha k}(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \eta, *), \qquad (25)$$

$$\overline{pu'_\alpha - p'u_\alpha} = P_\alpha(\bar{x}_1) p_\alpha(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \eta, *), \qquad (26)$$

where

$$\tilde{r}_1 = \frac{r_1}{l_s}, \text{ and } \tilde{r}_2 = \frac{r_2}{l_s}, \text{ and } \tilde{r}_3 = \frac{r_3}{l_s}. \qquad (27)$$

It is straightforward to show that these equations have self-similar solution of the form proposed if

$$B^{\alpha\alpha} \propto \frac{\epsilon_{\alpha\alpha} l_s^2}{\nu} \qquad (28)$$

and

$$T^{\alpha\alpha k} \propto P^\alpha \propto \epsilon_{\alpha\alpha} l_s. \qquad (29)$$

These scales are consistent with the scales develop by Kolomogorov for the second- and third-order structure functions; i.e.,

$$B^{\alpha\beta} \propto u_k^2 \qquad (30)$$

and

$$T^{\alpha\beta k} \propto u_k^3 \qquad (31)$$

where $u_k = (\nu\epsilon)^{1/4}$ and $l_s = (\nu^3/\epsilon)^{1/4}$. The solution for the small-scale motions are also compatible with the solutions for the large-scale motions because the scales for the small-scale motions are not independent of the large-scale motions. Instead they determine by the energy transfer from the large-scale motions.

Dependence on Initial Conditions

It is clear from the previous analysis that the hypothesis that the large-scale motions in the far field of the plane jet evolve in an equilibrium or self-similar manner is consistent with the governing equations. This analysis can also be used to examine if there is a single universal set of large-scale structures. It is straightforward to show that the $\bar{\xi}$ dependence can be eliminated from all of the terms in the governing equations yielding governing equations for the self-similar solutions that only depend on similarity variables. There is, however, no choice for the similarity scales that will eliminate the growth rate parameter $d\delta/dx_1$ from all of the terms in these equations. Thus, the solutions for the two-point velocity correlation and the large-scale structures occurring in the flow depend on the growth rate of the jet.

The proposition that there is an universal solution is normally supported by noting that the similarity solutions are only formally valid for virtual sources and that these sources only have a single invariant parameter the momentum flux M_o in the infinite Reynolds number limit. George [4, 5] argued that this would not necessarily hold true for flows originating from finite sources because they could have a range of invariants. However, if the flows from finite sources asymptotically approach self-similar or equilibrium solutions that are only valid for a virtual source then the solutions for these finite sources must take on the characteristics of the solutions for the virtual source. Thus, the additional invariants for the finite sources only determine how the flow approaches the self-similar solution for a virtual source or effectively the non-equilibrium character of the flow.

The error in the argument that the flows have an universal solution is the implicit assumption that the virtual source for turbulent self-similar flows are a steady source characterized only by its momentum flux M_o. Although a steady virtual source is appropriate for laminar flows, it is not for turbulent flows because the flow would have to undergo transition at some point downstream of the source. Thus, the self-similar solution for a turbulent flow could not describe the flow over its entire evolution. The virtual source must be unsteady with the appropriate turbulence intensity, frequency spectra, probability density function that are solutions to the equations for a given growth rate. It would require an infinite number of invariants to fully characterize these sources and there should be an infinite number of different unsteady sources.

Representation of the Large-Scale Structures

The large-scale structures in turbulent flows can be characterized using a number of techniques including Lumley's Proper Orthogonal Decomposition [8]. In this approach the flow is characterize using functions that optimally represent the energy in the turbulent flow. The optimal functions in the homogeneous directions are Fourier modes and these directions can be transformed from the problem. The functions in non-homogeneous directions are solution to the integral eigenvalue problem given by

$$\int R_{i,j}(\cdot,\cdot')\phi_j(\cdot')d\cdot' = \lambda\phi_i(\cdot). \qquad (32)$$

The functions for semi-infinite non-homogeneous directions, such as the mean flow direction in the plane jet, can be solved using this approach if the integral is bounded.

The optimal functions for the far field of the plane jet can be deduced from the similarity solutions for the two-point velocity correlation tensor. It was shown here that the similarity solutions for these correlations are only a function of the separation distances in the transformed streamwise direction. Thus, POD functions for this transformed direction are Fourier modes. The POD functions for the far field of the plane jet are thus given by

$$\int \tilde{F}_{i,j}(\tilde{k}_1,\eta,\eta',\tilde{k}_3)\phi_j(\eta',\tilde{k}_1,\tilde{k}_3)d\eta' = \lambda\phi_i(\eta',\tilde{k}_1,\tilde{k}_3), \qquad (33)$$

where $\tilde{F}_{i,j}$ is the Fourier transform of the similarity solution for the two-point velocity correlation in the lateral direction and the transformed streamwise direction; i.e.,

$$\tilde{F}_{i,j}(\eta,\eta'\tilde{k}_1,\tilde{k}_3) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_{i,j}(\upsilon,\eta,\eta',\zeta)e^{i\tilde{k}_1\upsilon}e^{i\tilde{k}_3\zeta}d\upsilon d\zeta. \qquad (34)$$

There is not an equivalent definition for the spatial spectrum in physical variables.

The two-point velocity correlation in the mean flow direction is often not measured. Instead, the frequency spectrum is normally measured at a single downstream location. It can be shown that the two-point two-time correlation has a self-similar solution of the form

$$R_{i,j}(x_1,x_2,x_2',r_3',\tau) = Q^{i,j}(x_1)q_{i,j}(\eta,\eta',\zeta,\tilde{\tau}), \qquad (35)$$

where $\tau = t - t'$ and $\tilde{\tau} = \tau/(\delta/U_s)$. It follows that the Fourier transform of this correlation; i.e.,

$$S_{i,j}(x_1,x_2,x_2',k_3,f) = \frac{1}{(2\pi)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{i,j}(x_1,x_2,x_2',r_3,\tau)e^{i2\pi f\tau}e^{ik_3r_3}d\tau dr_3 \qquad (36)$$

can be written as

$$S_{i,j}(x_1,x_2,x_2',k_3,f) = Q^{i,j}(x_1)\frac{\delta}{U_s}\delta_3\tilde{S}_{i,j}(\eta,\eta',\tilde{k_3},\tilde{f}) \qquad (37)$$

where $\tilde{f} = f/(U_s/\delta)$ and $\tilde{S}_{i,j}$ has an definition analogous to S_{ij} except in similarity variables.

The POD modes in the jet in self-similar variables are solutions to the integral eigenvalue problem given by

$$\int \tilde{S}_{i,j}(\eta,\eta',\tilde{k_3},\tilde{f})\tilde{\phi}_j(\eta^p r)d\eta^p r = \tilde{\lambda}\tilde{\phi}_i(\eta). \qquad (38)$$

These self-similar POD modes can be related the POD modes in physical variables; i.e.,

$$\int S_{i,j}(x_2,x_2',k_3,f)\phi_j(x_2^p r)dx_2^p r = \lambda(f,k_3)\phi_i(x_2) \qquad (39)$$

by scaling relationships given by

$$\tilde{\phi}_i(\eta) = \frac{1}{\delta(x_1)^{1/2}}\tilde{\phi}_i(\eta) \text{ and } \tilde{\lambda} = \frac{U_s\ \lambda}{Q_{i,j}\delta^3} \qquad (40)$$

Gordeyev and Thomas[6] used equivalent scaling to show that the POD modes in the plane jet are self-similar.

Conclusion

It was shown that the governing equations for the two-point velocity correlation tensor have self-similar solutions in the far field of the plane jet indicating the large-scale structures in this flow are evolving in an equilibrium manner. This result can be used to show that the orthogonal function determined using Lumley's POD are self-similar and the functions in the transformed mean flow direction are Fourier modes.

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