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Breakdown of Inertial Waves in a Tube with Variable Cross-Section

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Abstract

In this study we describe a theoretical asymptotic model for large amplitude stationary inertial waves in an axially symmetric swirling flow of an ideal fluid in a circular tube with variable cross-section. Calculations are presented for the special, but important case when the upstream flow is uniform and always supercritical. We find that the breakdown of inertial waves in divergent tubes is extremely sensitive to the variations of the cross-section. The breakdown bubble is generally asymmetric and it may turn into the diverging tail, especially in a convergent part of the tube. Possible relevance of the calculated structures to the experimentally observed types of the breakdown is discussed.

Introduction

There have been numerous studies of inertial waves in axisymmetric swirling flows in tubes, largely motivated by the possible relationship between such waves and the phenomenon of vortex breakdown [2,6]. This highly non-linear phenomenon exhibits the extreme sensitivity to a wide range of external influences, and to date there is no uniformly accepted model for this phenomenon. Main problem noticed in the early studies in the straight tubes is the upstream influence leading to the formation of the separation zone at the upstream boundary. One way to resolve this difficulty is to induce the breakdown by a local contraction of the tube, so that the separation zone is observed in the lee past that contraction [1,4]. Another includes introduction of specific inlet conditions preventing upstream propagation of disturbances [9]. Sophisticated calculations including the vortex shedding from the vane generator and a careful modeling of the geometry of the experimental set up were reported recently [8]. In all mentioned studies the specific geometry of the tube was fixed and the influence of its variations on the flow patterns was not examined. In this study we are going to develop a theoretical model for the stationary patterns which may contain breakdown (separation zones). We consider the special, but important case when the upstream flow is nearly uniform with the aim of emphasizing the effects associated with the influence of the variation of the cross-section of the tube.

In the next section we will formulate the mathematical model and describe the asymptotic development. Then, several calculations will be presented for the case of a solid body rotation. Finally, we speculate about the possible relevance of our calculations to the experimentally observed features.

Formulation

We consider the axisymmetric, steady flow of an inviscid, non-diffusive swirling fluid of constant density. The equation for the stream function ψ is the Bragg-Hawthorne equation

$$\psi_{xx} + \psi_{rr} - \frac{1}{r}\psi_r + C(\psi)C'(\psi) = r^2 G(\psi). \quad (1)$$

Here the axial and radial velocity components are given by

$$u = \frac{\psi_r}{r}, \quad v = -\frac{\psi_x}{r}, \quad (2)$$

and the swirl velocity w is given by

$$rw = C(\psi). \quad (3)$$

Note that in contrast to [3] we here set the wave speed (c) to be zero, as we wish to construct solutions, which are steady in the reference frame of the topographic perturbation to the bounding circular tube wall. The boundary conditions are that $\psi = 0$ on the tube axis $r = 0$, and ψ = constant on the tube wall $r = a + q(x)$, where $q(x)$ is the topographic perturbation.

Next, as in [3] we introduce dimensionless variables, based on a typical axial velocity U_0 and the upstream tube radius a. Thus,

$$\psi' = \frac{2\psi}{U_0 a^2}, \; r' = \frac{r}{a}, \; x' = \frac{x}{a}. \quad (4)$$

Henceforth we shall omit the prime superscripts. Then, as in [3] we will assume that the inflow axial velocity is nearly uniform, and the inflow angular velocity is likewise nearly uniform. Thus,

$$u \to u_\infty = 1 + \kappa U(\xi), \quad \text{as} \quad x \to -\infty, \quad (5)$$

$$\frac{w}{r} \to \frac{w_\infty}{r} = \Omega_0 (1 + \sigma\Omega(\xi)), \quad as \quad x \to -\infty, \quad (6)$$

where, $\xi = r^2$ and κ, σ are small parameters. It follows that we may write

$$\psi = \xi + \kappa \int_0^\xi u(\hat{\xi})\, d\hat{\xi} + \phi, \quad (7)$$

where ϕ is thus the perturbed streamfunction relative to the upstream value of ψ. After some algebra, we find that

$$\phi_{xx} + 4\xi\phi_{\xi\xi} + \lambda^2\phi = F(\phi),$$

(8) where $\lambda = |2\Omega_0|$ and $F(\phi)$ is a nonlinear function of

ϕ, given explicitly in [3] in terms of the known functions $U(\xi)$ and $\Omega(\xi)$. Here $\frac{1}{2}\lambda = |\Omega_0|$ is the swirl number which measures the ratio of the swirl velocity at the tube wall to the axial velocity. The boundary conditions at the tube axis and wall are that $\phi = 0$ at $r = 0$, and that

$$\phi+\xi-1+\kappa\int_1^{\xi} U(\hat{\xi})d\hat{\xi}=0 \text{ at } r=1+hq(x), \qquad (9)$$

Here we have introduced another small parameter, h, to measure the radial extent of the topographic wall perturbation.

Then, again as in [3], we allow for the possibility that there may be a recirculation zone located on the axis of the tube. This occurs when

$$1 + \phi_{\xi} = 0 \quad \text{at} \quad \xi = 0 \qquad (10)$$

which defines a critical wave amplitude. For waves with larger amplitudes, we suppose that there is a separation zone whose boundary is given by

$$r = \eta(x) \qquad (11)$$

where $\eta = 0$ outside the domain $x_- < x < x_+$ say. Inside the separation zone, the governing equation is again (1) but the circulation function $C(\psi)$ and the vorticity function $G(\psi)$ must be determined anew. On the boundary (11) of the separation zone, $\psi = 0$ as it is a streamline, and continuity of pressure leads to continuity of ψ_{ξ}.

Asymptotic development

The x-domain is divided into two parts, an outer zone where we seek solutions whose axial length scale is ε^{-1} where $0 < \varepsilon << 1$, and an inner zone $(x_- < x < x_+)$ which includes the separation zone and where the axial length scale is β^{-1} where $\beta = \varepsilon^{1/2}$. In the outer zone, we introduce the long axial variable

$$X = \varepsilon x, \qquad (12)$$

and then, as in [3], seek an asymptotic expansion of the form

$$\phi = \phi^{(0)}(X, \xi) + \varepsilon^2\phi^{(1)}(X, \xi)+..., \qquad (13a)$$

$$\lambda = \lambda^{(0)} + \varepsilon^2\lambda^{(1)} +.... \qquad (13b)$$

Here we assume that κ, σ, h are $O(\varepsilon^2)$, and so put $\kappa = \hat{\kappa}\varepsilon^2$, $\sigma = \hat{\sigma}\varepsilon^2$ and $h = \hat{h}\varepsilon^2$. Note that here, in contrast to [3], we expand the swirl number (i.e. $\frac{1}{2}\lambda$) directly. The analysis now proceeds as in [3], the only essential new feature being the inclusion of the topographic perturbation in the tube wall boundary condition. Hence, at the lowest order, we get

$$\phi^{(0)} = A(X)W(\xi), \qquad (14a)$$

where

$$W(\xi) = rJ_1(\lambda^{(0)}r), \qquad (14b)$$

and

$$J_1(\lambda^{(0)}) = 0. \qquad (14c)$$

For the lowest mode $\lambda^{(0)} = 3.83$, and we shall consider only this mode henceforth. At the next order, a compatibility condition is applied to the equation for $\phi^{(1)}$ which yields the required amplitude equation

$$-A_{XX} + \Delta A + M(A) + \hat{h}q(J_1 - AJ_2) = 0. \qquad (15)$$

Here, the nonlinear function $M(A)$ is identical to that given by [3] (with $c^{(0)}$ there set equal to zero), while here $\Delta = 2\lambda^{(0)}\lambda^{(1)}$. Thus $\Delta > 0 (< 0)$ corresponds to an increase (decrease) in the swirl number respectively. The constants J_1, J_2 are defined by

$$J_1 = -\frac{4\lambda^{(0)}}{J_0(\lambda^{(0)})}, \quad J_2 = -2\lambda^{(0)2}, \qquad (16)$$

we note that $J_1 \approx 38.1 (>0)$ and $J_2 \approx -29.4 (<0)$.

An important issue now arises, namely that in (2.15) it has been implicitly assumed that $q = q(X)$, i.e. the topography varies on the outer length scale. Clearly, this is not necessarily the case, and indeed it is also interesting in the present context to assume that the topography varies on the inner length scale, that is, $q = q(z)$ where $z = \beta x$ and $\beta = \varepsilon^{1/2}$, and is non-zero only in the domain $0 < z < z_T$ $(z_T = \beta x_T)$.

In the inner zone, the analysis again follows that of [3] very closely. Thus, we suppose that the inner length scale is β^{-1} where $\beta = \varepsilon^{1/2}$, and that the size of the separation zone $O(\beta)$. Hence we put

$$\eta = \delta f(z), \quad \text{and} \quad z = \beta x, \qquad (17)$$

where the domain of the inner zone is defined by $z_- < z < z_+$, noting that $f(z) = 0$ outside this domain. The location of the inner zone boundary is determined by the criterion that at $z = z_{\pm}$, or equivalently, at $X = X_{\pm}$ where $X_{\pm} = \beta z_{\pm}$, the wave amplitude A reaches the critical value A_* where there is incipient flow reversal, that is (10) is satisfied. Using the leading order expression $\phi^{(0)}$ (14a) for ϕ, we find that

$$A_* = -2/\lambda^{(0)}, \tag{18}$$

and that, as already anticipated, the flow reversal occurs on the tube axis. Note that $A_* \approx -0.52 < 0$, and so waves with separation zones are waves of "elevation", i.e. the streamlines are displaced in the direction of r increasing. In the inner zone, $A(z)$ is close to this critical amplitude, and so we put

$$A(z) = A_* - \mu B(z),$$

(19)

where an optimal balance of small parameters requires that $\mu = \varepsilon$ and $\delta = \varepsilon^{1/2}$. However, we find it convenient to retain μ and ε as independent small parameters, albeit of the same order, as the ratio measures the magnitude of the separation zone vis-à-vis the magnitude of the outer length scale. Proceeding as in [3] we find that B satisfies the equation

$$B_{zz} + \Delta A_* + M(A_*) - \nu B^2 + \hat{h} q J_* = 0, \tag{20a}$$

where

$$J_* = J_1 - A_* J_2, \tag{20b}$$

and

$$\nu = \frac{\lambda^{(0)}}{J_0^2(\lambda^{(0)})} \frac{\mu^2}{\varepsilon^2}. \tag{20c}$$

Note that $J_* \approx 22.7 > 0$. Also, the boundary of the separation zone is given by

$$f^2 = \frac{2}{\lambda^{(0)}} \frac{\mu}{\varepsilon} B. \tag{21}$$

Here, we define the small parameter μ as a measure of the maximum size, A_m, of the wave amplitude, that is $A_m = A_* - \mu$. Thus, we must have $0 < B < 1$. Then the amplitude equation (2.20a) is to be solved subject to the matching conditions,

$$B = 0 \quad \text{and} \quad A_X = \mp \sqrt{\mu} B_z, \quad \text{at} \quad z = z_\pm. \tag{22}$$

Finally, the analysis inside the separation zone proceed as in [3], and hence we can conclude that the separation zone is nearly stagnant.

Specific case of the uniform flow.

In this section we consider the important special case when the upstream inflow conditions are uniform axial and uniform angular velocity. The radial component of the velocity is set to be zero. Thus the functions $U(\xi)$ and $\Omega(\xi)$ in (2.5), (2.6) are zero, and nonlinearity function $M(A)$ in (2.15) and (2.20) is zero. That is the only nonlinearity in the problem arises due to the flow over the separation zone. We shall examine the case when $A = 0$, $A_x = 0$, and $q = 0$ for $x \le 0$.

In the first set of calculations in the divergent tube (figure 1) with the profile $q(x) = -0.055 + 0.025x$, we observed that the separation zone moves upstream and its size increases as swirl Ω_0 increases (Δ increases). Note that the separation zone in a diverging tube has both the radial and the axial size finite even for the maximum amplitude contrary to the infinite separation zone in an ideal fluid in a straight tube [3,5]. The middle part of the bubble of the maximum amplitude presented in figure1 is tilted and the back of the separation zone is steeper than its front, i.e. the separation zone is essentially asymmetric. This is consistent with numerous observations of the laminar breakdown [2,6,7].

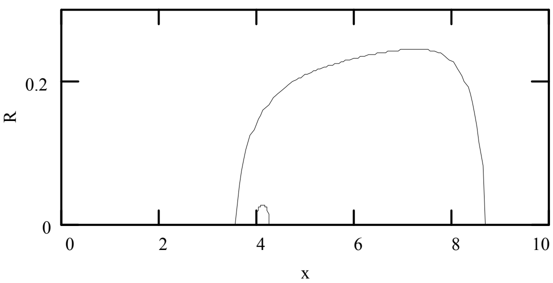

Figure 1. Profiles of the separation zone in a diverging tube for different swirls, Δ=–0.008 (corresponds to smaller swirl) for the small bubble, and Δ=0.0199 for the large bubble of nearly maximum possible amplitude.

In figure 2 we illustrate that to observe the onset of the stagnation point at some fixed position, the swirl should be decreased as the angles of the divergence are sharpened. Qualitatively similar effect was observed by Sarpkaya [7], although velocity profiles in the approaching vortex in his experiments differ significantly from the uniform flow examined here.

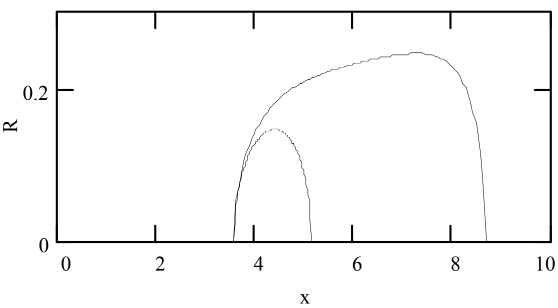

Figure 2. Profiles of the separation zone occurring at the same position at the in tubes with different angles of divergence, Δ and the angle of divergence are –0.1233 and 1.189° for the small bubble; 0.0199 and 1.432° for the large bubble respectively.

It is pertinent to note that the flow is supercritical at the beginning of divergence for all presented Δ as solutions of (15) are exponential functions there, but at some cross-section solutions turns to be subcritical. Leibovich discussed similar behavior in [6].

In the case of diverging tube our calculations show that the streamlines significantly converge just several bubble diameters downstream of the back side of the separation zone. It may lead to instabilities in a newly formed jet-like vortex or even to the onset of another separation zone near the boundary of the tube. As these instabilities may violate our initial assumption that the flow is axisymmetric, our model is not applicable far downstream the separation zone. When the divergence occurs only at a finite interval, the convergence of the streamlines past the separation is relatively mild (see figure 3). In this figure the divergence mainly affect the flow before the stagnation occurs, thus separation zone is fairly symmetric.

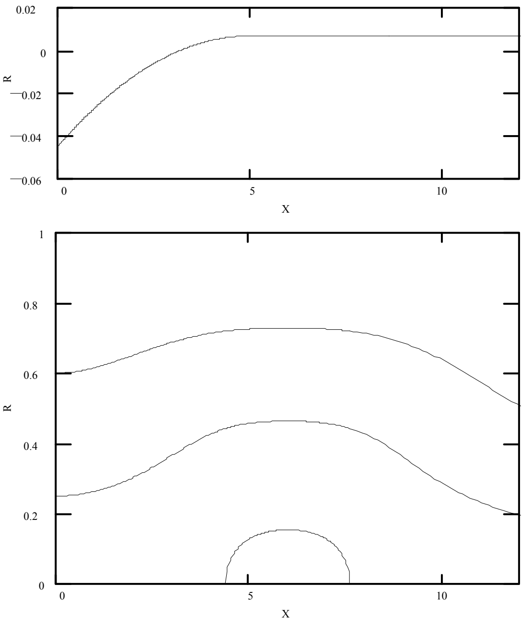

Figure 3. Profile of the tube and the corresponding flow pattern in the tube with a finite interval of mild divergence. Δ= –0.1025.

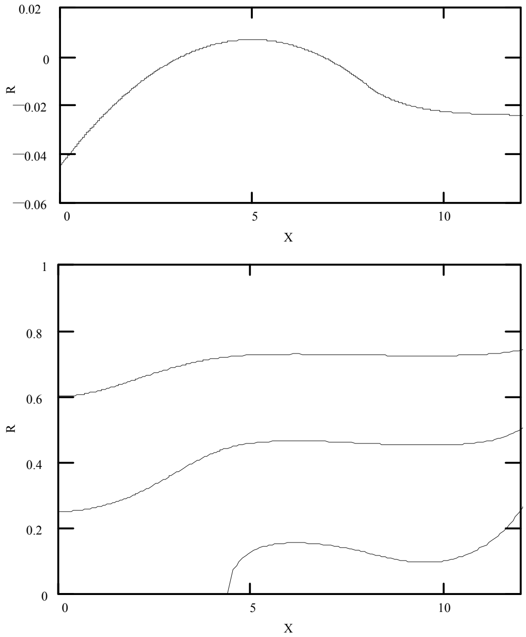

Figure 4. Profile of the tube and the corresponding flow pattern in the tube with both divergent and convergent sections. Δ= –0.1025 the same as for the flow shown in the previous figure.

When the divergent section of the tube is followed by the interval of convergence, we found that the separation zone may consist of the bubble-like part, which finally joins the divergent tail. The flow pattern for this case is shown in figure 4. Note that swirls in the approaching vortex are the same in figures 3 and 4. The flow structure changes entirely just due to the slight convergence of the tube. The shape of the separation zone analogous to that presented in figure 4 was observed by Sarpkaya [7] for the case of turbulent regime in the divergent tube. A possible explanation of this coincidence may be the following. For the turbulent regimes studied by Sarpkaya [7], the turbulent boundary layer develops much more rapidly than in the classical works on the laminar breakdown [6]. Finally, in a certain cross-section, the convergence of the duct due to the growing turbulent boundary layer may compete and even prevail the actual divergence of the tube thus effectively creating the profile of the confining boundary with both of divergent (initial) and convergent sections similar to that presented in figure 4. Sarpkaya [7] reported that the bubble structure first rejoins the tail when the value of the fluctuation velocity relative to the axial velocity in the approaching vortex is 2.3%. Therefore, the tangent of the angle of convergence due to the growing boundary layer is of the order of 0.023 which is close to the actual tangent 0.025 of the divergence of the tube itself .

Conclusions

In this paper we presented a new asymptotic theory for the inertial waves in swirling axisymmetric flows. Our model is set to account the effects due to the separation zones appeared in a flow. Calculations are presented for the special, but important case when the upstream flow is uniform. We found that the breakdown of inertial waves in divergent tubes is extremely sensitive to the variations of the cross-section of the tube. The breakdown bubble is generally asymmetric and it may turn into the diverging tail, especially in a convergent part of the tube.

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