14th Australasian Fluid Mechanics Conference
Adelaide University, Adelaide, Australia
10-14 December 2001

Pressure Field Calculation in Flow Simulation by Discrete Vortex Method

K.L. Lai and M.K. Bull

Department of Mechanical Engineering
University of Adelaide, Adelaide, South Australia, 5005 AUSTRALIA

Abstract

In numerical simulations of fluid flow by discrete-vortex methods, the natural processes of vorticity creation at solid boundaries and vorticity evolution in the flow domain are directly modelled. The governing equations are formulated in terms of vorticity, with the pressure terms eliminated. The calculations then yield directly the evolution of the vorticity field. From the vorticity field, streamlines and pressure fields are readily obtainable. The derivation of the pressure field involves evaluation of the time-rate of change of the velocity potential resulting from variation with time of the surface-vorticity on solid boundaries. The velocity potential, and hence the pressure, can formally have physically-inadmissible multiple values. Numerical procedures for the derivation of the pressure field from the vorticity field are detailed, which prevent the occurrence of multiple values or discontinuities in the calculated pressure.

Introduction

A numerical scheme, based on the discrete vortex and surface-vorticity boundary-integral methods, has been developed to calculate the time-dependent, two-dimensional flow over arrays of bluff bodies. Calculations yield directly the evolution of vorticity field in terms of the distribution of discrete vortices in the flow domain and of surface-vorticity on solid boundaries. Flow development is derived from the evolution of the vorticity field. Variation of stagnation pressure in the flow is induced by two mechanisms: the movement of vortices in the flow, and the time-rate of change of surface vorticity on solid boundaries. Calculation of the stagnation-pressure variation induced by surface-vorticity elements involves the evaluation of the velocity potential associated with the surface-vorticity elements. However, the multi-valued nature of the velocity potential is not consistent with the continuous and single-valued nature of the pressure field. These problems are addressed and the derivation of numerical procedures for their solution presented.

Theoretical Background

In the absence of external force, the Navier-Stokes equation for an incompressible flow can be expressed in the form

$$\partial_t \mathbf{u} - \mathbf{u} \text{ x } \omega = -\frac{1}{\rho}\nabla \mathbf{p}_0 + \nu\nabla^2\mathbf{u} \,, \tag{1}$$

or, in terms of vorticity, as

$$\partial_t \omega + \mathbf{u} \bullet \nabla\omega = \omega \bullet \nabla\omega + \nu\nabla^2\omega \,, \tag{2}$$

where $\mathbf{u}$ is fluid velocity vector, $\boldsymbol{\omega}=\nabla\times\mathbf{u}$ the vorticity, t time and ν the fluid viscosity. Implementation of Eq. (2) in vortex methods for calculating two-dimensional flow involves splitting the advective-diffusive motion in each time step into two fractional steps [1]: convection ($\partial_t\boldsymbol{\omega}+\mathbf{u}\cdot\nabla\boldsymbol{\omega}$) and diffusion $\nu\nabla^2\boldsymbol{\omega}$ of vorticity (vortex stretching $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}$ can occur only in three-dimensional flow). In a system consisting of a solid body which is represented by M panels, the boundary condition of zero-velocity at solid boundaries is satisfied (at collocation points of the panels) by creating a vorticity sheet which is approximated by a set of two-dimensional surface-vorticity elements with strengths γ_n, n = 1,2,...M, each element being equivalent to a discrete vortex of strength $\gamma_n\Delta s_n$ (where Δs_n is the length of the element). For the satisfaction of the closure of the surface-pressure distribution on the solid body, the condition of zero-circulation around the body contour is explicitly expressed in the governing equation for vorticity creation. Thus, the distribution of surface-vorticity created on the body satisfies the condition $\Sigma\gamma_n\Delta s_n = 0$. The vorticity that is created at solid boundaries and shed into the flow is represented by a set of discrete-vortices, with circulation $\Delta\Gamma_j$, j = 1,2,...Z. Thus mathematical modelling of bluff body flows by the method of discrete vortices involves determination of the circulations of the bound discrete vortices and the evolution of the vortices in the flow: a solution is to be obtained for the vorticity distribution as a function of space and time.

By using the vector identity $\nabla^2\boldsymbol{u}\equiv\nabla(\nabla\cdot\boldsymbol{u})-\nabla\times(\nabla\times\boldsymbol{u})$, Eq. (1) can be rewritten as

$$\frac{1}{\rho}\nabla\mathbf{p}_0 = (\mathbf{u} - \nu\,\nabla) \text{ x } \omega - \frac{\partial\mathbf{u}}{\partial\mathbf{t}} \quad . \tag{3}$$

Given a vorticity field $\boldsymbol{\omega}$ and its evolution with time, the velocity field $\mathbf{u}$ and its time derivatives $\partial_t\mathbf{u}$ at any point can be obtained, and hence Eq. (3) can be integrated to give the stagnation pressure p_o at any point. In the vortex method, with the continuous vorticity field represented by a system of discrete vortices, the vorticity $\boldsymbol{\omega} = 0$ in the flow domain, except at the vortex singularities. Eq. (3) therefore reduces to

$$\frac{1}{\rho}\nabla\mathbf{p}_0 = -\frac{\partial\mathbf{u}}{\partial\mathbf{t}} \;. \tag{4}$$

Thus, ∇p_o can be obtained by evaluation of the time derivative of velocity field $\partial_t\mathbf{u}$. Given the vorticity field at two distinct time instants, say $\boldsymbol{\omega}(t)$ and $\boldsymbol{\omega}(t+\Delta t)$, we are able to construct the associated velocity fields $\mathbf{u}(t)$ and $\mathbf{u}(t+\Delta t)$, and hence the time derivative $\partial_t\mathbf{u}$, by finite difference methods. In the present numerical scheme, with the motion of discrete vortices being tracked, the evolution of the velocity field is a function of the motion of the discrete vortices, and $\partial_t\mathbf{u}$ can be derived directly from the vortex motion. In the presence of a uniform free-stream $\mathbf{U}_\infty$, a system of Z discrete vortices $\Delta\Gamma_j$ at position (x_j, y_j) in the flow domain and a distribution of surface vorticity $\gamma\Delta s$ on solid boundaries, the velocity $\mathbf{u}$ at point (x, y) can be expressed as

$$\begin{aligned}\mathbf{u} = \mathbf{U}_\infty &+ \sum_{j=1}^{Z}\frac{\Delta\Gamma_j}{2\pi r_j^2}\left[(\mathbf{y}-\mathbf{y_j}),-(\mathbf{x}-\mathbf{x_j})\right] \\ &+ \sum_{m=1}^{M}\frac{\gamma_m\Delta s_m}{2\pi r_m^2}\left[(\mathbf{y}-\mathbf{y_m}),-(\mathbf{x}-\mathbf{x_m})\right]\end{aligned} \tag{5}$$

where $r_j^2 = (x-x_j)^2 + (y-y_j)^2$, $r_m^2 = (x-x_m)^2 + (y-y_m)^2$, ($x_m$, y_m) denotes the coordinates of the collocation point on panel segment s_m, and γ_m is the strength of the surface-vorticity sheet corresponding to the system of Z discrete vortices at their current positions. Note that Eq. (5) describes a velocity field in which the boundary conditions on solid boundaries are satisfied.

Numerical Scheme

It is pertinent to introduce the numerical scheme before differentiation of Eq. (5) is carried out to obtain $\partial_t\mathbf{u}$. Consider a flow started impulsively from rest at t = 0 and its subsequent development calculated using a timestep Δt. The numerical procedures in each timestep involve convection and diffusion of discrete vortices, creation of surface vorticity on solid surfaces and its subsequent introduction into the flow as nascent discrete-vortices. The beginning of a time step is taken to be the instant just after shedding of surface vorticity γ_m into the flow domain. The entire vorticity field $\boldsymbol{\omega}(t)$ then consists of a system of Z discrete vortices at positions (x^k, y^k), where Z includes the nascent vortices equivalent to the surface vorticity sheet which have just been shed from the surface. At this particular moment, the surface vorticity sheet has vanished, i.e. $\gamma_m = 0$. However, the boundary conditions are still satisfied because the nascent vortices are effectively equivalent to the surface vorticity sheet.

Evolution of this system of Z discrete vortices by convection and diffusion, over the duration of a timestep Δt, brings them to their new positions (x^{k+1}, y^{k+1}). Corresponding to these new vortex positions (x^{k+1}, y^{k+1}), a new surface vorticity sheet is created to restore the boundary condition at solid boundaries. It is then conceptually correct to consider the surface vorticity sheet being changed from zero to a finite non-zero value over the duration of a timestep Δt. As a result of the motion of these Z discrete vortices and the creation of a new surface vorticity sheet, the entire vorticity field has evolved from $\boldsymbol{\omega}(t)$ to $\boldsymbol{\omega}(t+\Delta t)$, and the appropriate boundary conditions have been restored at solid boundaries in readiness for the next time step. At this point, the final instant in the time step, the newly created surface vorticity sheet is shed into the flow domain as nascent vortices at the very beginning of the next time step. It is at this moment that the system of Z discrete vortices is increased by the number of nascent vortices; Z remains constant throughout the remainder of the timestep.

It should be noted that the creation of surface vorticity on solid boundaries is associated with the motion of discrete vortices; and that the surface vorticity γ_m is the only quantity of vorticity that changes with time; the circulation of discrete vortices $\Delta\Gamma_j$ in the flow field remains constant, i.e. $\partial_t(\Delta\Gamma_j)=0$.

Calculation of Pressure field

Taking the time derivative of the velocity field of Eq. (5), we have

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \frac{1}{2\pi}\sum_{\mathbf{j}=1}^{\mathbf{Z}} \Delta\Gamma_{\mathbf{j}} \frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{1}{\mathbf{r_j^2}}\left[(\mathbf{y}-\mathbf{y_j}),-(\mathbf{x}-\mathbf{x_j})\right]\right) + \frac{1}{2\pi}\sum_{\mathbf{m}=1}^{\mathbf{M}} \frac{\mathbf{d}\gamma_{\mathbf{m}}}{\mathbf{dt}}\frac{\Delta \mathbf{s_m}}{\mathbf{r_m^2}}\left[(\mathbf{y}-\mathbf{y_m}),-(\mathbf{x}-\mathbf{x_m})\right]. \tag{6}$$

Let $\mathbf{V}_j\ [v_{xj}, v_{yj}] = [dx_j/dt, dy_j/dt]$ denote the transport velocity of discrete vortex $\Delta\Gamma_j$ from its initial position (x^k, y^k) at the beginning of a timestep to the final position (x^{k+1}, y^{k+1}) at the end of that timestep. Note that the transport velocity $\mathbf{V}_j$ is the result of both convection and diffusion and is generally different from the local flow velocity, noted by (u^k, v^k), at vortex position (x^k, y^k).

Substitution of Eq. (6) in Eq. (4) gives

$$-\frac{1}{\rho}\nabla \mathbf{p}_0 = \sum_{\mathbf{m}=1}^{\mathbf{M}} \frac{\mathbf{d}\gamma_{\mathbf{m}}}{\mathbf{dt}}\frac{\Delta \mathbf{s_m}}{2\pi \mathbf{r_m^2}}\left[(\mathbf{y}-\mathbf{y_m}),-(\mathbf{x}-\mathbf{x_m})\right] + \sum_{\mathbf{j}=1}^{\mathbf{Z}} \frac{\Delta\Gamma_{\mathbf{j}}}{2\pi \mathbf{r_j^4}}\begin{bmatrix} 2(\mathbf{x}-\mathbf{x_j})(\mathbf{y}-\mathbf{y_j})\mathbf{v_{xj}} - \left[\mathbf{r_j^2}-(\mathbf{y}-\mathbf{y_j})^2\right]\mathbf{v_{yj}}, \\ \left[\mathbf{r_j^2}-2(\mathbf{x}-\mathbf{x_j})^2\right]\mathbf{v_{xj}} - 2(\mathbf{x}-\mathbf{x_j})(\mathbf{y}-\mathbf{y_j})\mathbf{v_{yj}} \end{bmatrix} \tag{7}$$

With knowledge of the derivative $\partial_t\mathbf{u}$ and hence the pressure gradient ∇p_o, one may compute p_o at any point by integrating Eq. (7) numerically by the finite difference methods. In this case, it is vital to form a finite difference mesh large enough to establish known boundary conditions (i.e. stagnation pressure at infinity $p_o = p_\infty + ½\rho U_\infty^2$) and to use a finite difference step size small enough to reveal detail in regions of high pressure gradient, as well as to minimise numerical error. These constraints on finite difference methods make direct numerical integration of Eq. (7) very inefficient in terms of computing time.

Alternatively, integration of Eq. (7) can be carried out analytically to give p_o. The two sums on the right-hand side of Eq. (7) relate the stagnation pressure distribution to the change of surface vorticity with time and to the transport of vorticity in the flow, and will be treated separately. Integrating the second summation of Eq. (7) and introducing the term p_{ov} to represent the stagnation pressure induced by the moving vortices, we have

$$\begin{aligned}\frac{\mathbf{p_{ov}}}{\rho} &= \frac{1}{\rho}\int \nabla \mathbf{p_{ov}} \bullet \mathbf{ds} = \sum_{\mathbf{j}=1}^{\mathbf{Z}} \frac{\Delta\Gamma_{\mathbf{j}}}{2\pi \mathbf{r}^2}(\mathbf{y}\,\mathbf{v_{xj}} - \mathbf{x}\,\mathbf{v_{yj}}) \\ &= \sum_{\mathbf{j}=1}^{\mathbf{Z}} \mathbf{U_{ij}} \bullet \mathbf{V_j}\end{aligned} \tag{8}$$

where $\mathbf{U}_{ij}$ is the velocity at i induced by a vortex $\Delta\Gamma_j$ at j. This result, Eq. (8), is identical to the expression presented by Porthouse (1983) who derived the stagnation pressure from the term $(\mathbf{u}-\nu\nabla)\times\boldsymbol{\omega}$, interpreted as vortex flux by Porthouse. Note that the pressure field p_o is conservative. This allows us to use Eq. (8) to evaluate the stagnation pressure induced by the moving vortices.

For the variation in stagnation pressure $p_{o\gamma}$ induced by the changing surface vorticity, the pressure gradient is

$$-\frac{1}{\rho}\nabla \mathbf{p_{o\gamma}} = \sum_{\mathbf{m}=1}^{\mathbf{M}} \frac{\mathbf{d}\gamma_{\mathbf{m}}}{\mathbf{dt}}\frac{\Delta \mathbf{s_m}}{2\pi \mathbf{r_m^2}}\left[(\mathbf{y}-\mathbf{y_m}),-(\mathbf{x}-\mathbf{x_m})\right] \ . \tag{9}$$

The stagnation pressure field $p_{o\gamma}$ is conservative and the integration of $\nabla p_{o\gamma}$ between any two distinct points is therefore independent of the path between the points. The integration gives $p_{o\gamma}$ as

$$\begin{aligned}\frac{\mathbf{p_{o\gamma}}}{\rho} &= \frac{1}{2\pi}\sum_{\mathbf{m}=1}^{\mathbf{M}} \frac{\mathbf{d}\gamma_{\mathbf{m}}}{\mathbf{dt}}\Delta \mathbf{s_m}\theta_{\mathbf{m}} \\ &= \sum_{\mathbf{m}=1}^{\mathbf{M}} \frac{-\phi_{\mathbf{v\,m}}}{\Delta \mathbf{t}}\end{aligned} \ , \tag{10}$$

where γ_m is the surface vorticity created on solid boundaries during time Δt, $\theta_m=\tan^{-1}(y-y_m)/(x-x_m)$ the angle made at the pivotal point on segment s_m by a point at (x, y), and $\varphi_v = -\Delta\Gamma\theta/2\pi$ the velocity potential of a discrete point-vortex $\Delta\Gamma$. The term $d\gamma_m/dt$ in Eq. (10) is approximated by $\gamma_m/\Delta t$ because the magnitude of the surface vorticity sheet changes from 0 to γ_m over a timestep Δt. It can be seen that $p_{o\gamma}$ given by Eq. (10), is the sum of the time rates of change of the velocity potentials of all the vortex sheet elements on solid bodies.

By the principle of superposition, the two potential-flow pressure fields p_{ov} and $p_{o\gamma}$ can be added to give the stagnation pressure p_o ($= p_{ov} + p_{o\gamma}$) which is a solution to the Navier-Stokes equation. However, before Eq. (10) is evaluated for $p_{o\gamma}$, consideration of the mathematical nature of the velocity potential φ_v of a discrete vortex is pertinent. Being a function of the angular displacement θ, the velocity potential φ_v of a point vortex $\Delta\Gamma$ is a multi-valued scalar function: as θ increases from zero, φ_v also increases until at $\theta = 2\pi$ it becomes $\varphi_v = \Delta\Gamma$. Another circuit round the vortex increases φ_v by another $\Delta\Gamma$. A multi-valued function is not consistent with the single-valued and continuous nature of the pressure field. Therefore, it is necessary to clarify the calculation of $p_{o\gamma}$ from the velocity potential of a vortex.

It is instructive to consider the particular case of an isolated discrete vortex $\Delta\Gamma$ in an unbounded fluid which induces a circulating motion in the fluid around it. The fluid is at rest except in so far as it is disturbed by the vortex; in particular the velocity and pressure tend to zero at great distances from the vortex. Assume that we are able to increase the circulation of this vortex by $d\Delta\Gamma$ during a time increment of dt. Use of Eq. (10) to compute the pressure distribution associated with $d\Delta\Gamma/dt$, through the term $d\varphi_v/dt$, would imply a multiplicity of values for the pressure and also a constant non-zero finite total pressure, $p_o \neq 0$, along any radial direction even at infinity; both these effects are physically impossible. Prandtl and Tietjens (1934, Art. 71) discuss this issue and conclude that if a motion with circulation exists it will persist, but such motion cannot be produced from rest. It is physically impossible to change the total circulation in the fluid without temporarily upsetting the continuity of the fluid as, for example, by the insertion of a rigid body into the fluid. This conclusion is consistent with Kelvin's circulation theorem which states that the circulation in a flow remains constant with time.

In a real flow, and also in the present numerical model, equal amounts of vorticity of opposite sign are generated on the body surface, so that the net circulation created on the body is equal to zero. Despite the change in circulation ($\gamma\Delta s$) with time on each segment of the body, the time rate of change of circulation on the whole body remains unchanged and equal to zero. An observer distant from the body will not experience any change in velocity or in the pressure field due to the change of surface vorticity on the body. Provided the condition of zero circulation generated around a body contour is satisfied, Eq. (10) is valid to predict the stagnation pressure field associated with the time rate of change of surface vorticity on the body. In the present numerical scheme, the condition of zero circulation around a body contour has been satisfied by the enforcement of $\Sigma\gamma\Delta s = 0$, and hence the prerequisite condition for the evaluation of $p_{o\gamma}$ from Eq. (10) is fulfilled. The practical problem remaining is the determination of θ values so as to avoid discontinuities and multiple values of φ_v.

Evaluation of Eq. (10) for $p_{o\gamma}$ at a point (x, y) requires determination of the angles θ_m (m = 1,2,...M) which the lines joining the collocation points on the body to that point make with the reference datum. Although the intrinsic multi-valuedness of φ_v could be eliminated by setting the range of θ to be 0 to 2π, φ_v then experiences a discontinuity as θ moves across the datum axis, where its value jumps from 0 to 2π; and when a point lies on the datum axis, φ_v has an indeterminate value between 0 or 2π. Because it is the entire vortex sheet on a body that determines the pressure distribution induced by the creation of surface vorticity, the pressure given by Eq. (10) will be correct only if θ_m is finite and well-defined and its variation around the body contour is smooth and continuous. Evidently, when these conditions are satisfied, the sum in Eq. (10) becomes independent of the selection of datum axis. We must therefore select a proper datum axis for measuring θ, to ensure a smooth and continuous variation in θ around the body contour, and its orientation will be determined by the geometric relationship between a particular point and the body.

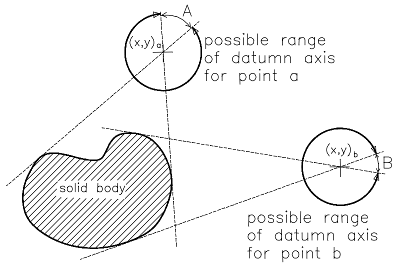


Figure 1. Determination of the reference datum axis for the evaluation of θ_m.

Consider a point $(x, y)_a$ in the flow field as shown in Figure 1. The angle between the tangents from it to the solid body is A. All the lines joining the pivotal points on the body and point $(x, y)_a$ must lie between these tangent lines. If the datum axis has an orientation which lies inside angle A, the values of θ_m measured from this datum axis will fall in the vicinity of the discontinuity. However, when the datum axis is rotated to such an angular position that it lies outside angle A, the values of θ_m measured from the new datum axis will have a smooth and continuous variation. Any angular position outside the angle A can be used as a proper orientation for the datum axis for point $(x, y)_a$.

Since the orientation of the datum axis for a particular point is determined by its geometric relationship with the body, it has to be determined independently for each point (for each body in multi-body systems). Figure 1 shows the range of angular position that is possible as an orientation for the datum axis for two distinct points in the flow field: (360°-A) for point $(x, y)_a$ and (360°-B) for point $(x, y)_b$.

Adding the contributions to the stagnation pressure from the uniform main stream, the moving vortices, and the creation of surface vorticity, we obtain the pressure coefficient at any point in the flow field as

$$\mathbf{C_p} = 1 + 2\left[\sum_{\mathbf{j}=1}^{\mathbf{Z}} \mathbf{U_{ij}} \bullet \mathbf{V_j} + \sum_{\mathbf{m}=1}^{\mathbf{M}} \frac{\gamma_\mathbf{m} \Delta \mathbf{s_m}}{2\,\pi\,\Delta \mathbf{t}} \theta_\mathbf{m}\right] - \mathbf{u}^2 \quad , \qquad (11)$$

where the pressure coefficient is defined as $C_p = (p - p_\infty)/\frac{1}{2}\rho U_\infty^2$, and the other symbols represent non-dimensional quantities. The pressure coefficient given by Eq. (11) is in fact an analytical solution to the contour integral of pressure gradient given by Eq. (7), along a path, excluding the body contour, from infinity to a point in the flow domain or on the body contour.

Application of Numerical Scheme

The numerical scheme developed has been applied to flow over a tandem array of two square cylinders, with gap-to-thickness ratio G = 2, at Reynolds number $Re_h = 500$ (based on plate thickness h). The boundary of each of the cylinders is represented by 80 equal straight panels. The flow is started impulsively from rest at t = 0, and its subsequent development calculated at times t (normalised by U_∞ and h), advancing by intervals of $\Delta t = 0.02$.

Discrete-vortex distributions yielded directly by the calculation, for two time instants of the fully-developed flow are shown in Figure 2. The corresponding streamlines and instantaneous pressure fields are presented in Figures 3 and 4 respectively. The values shown in Figure 4 are coefficients of static pressure obtained by using Eq. (11).

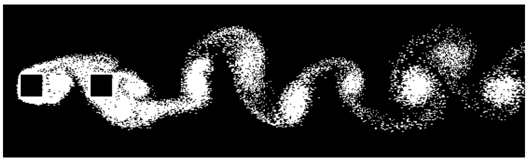


Figure 2. Discrete vortex distribution for flow over a tandem array of square cylinders with gap G = 2; t = 42

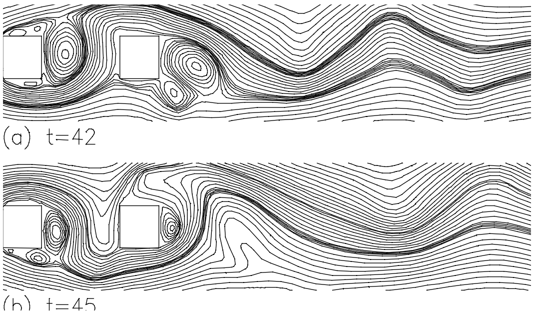
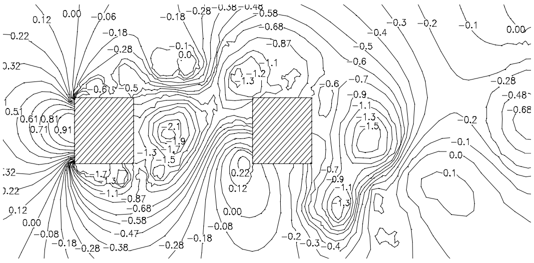
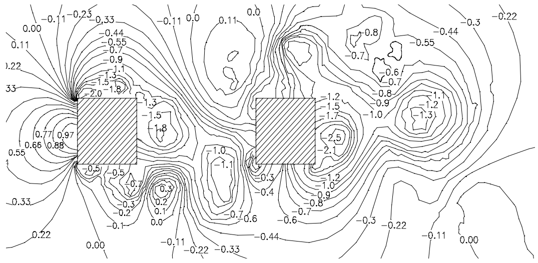


Figure 3. Streamline patterns: (a) t = 42; (b) t = 45.



(a) t = 42



(b) t = 45

Figure 4. Pressure contours: (a) t = 42; (b) t = 45. Values shown are coefficients of static pressure.

Conclusion

A numerical procedure for calculating the pressure field in flow simulation by discrete-vortex methods has been developed. By tracking the position and motion of discrete vortices and the rate of change of surface-vorticity at solid surfaces, the pressure field at any general point in the flow field and also on the solid surfaces can be calculated. The problems of multi-valuedness and discontinuities in the calculated pressure, associated with the velocity potential of vortices, are overcome by relating the surface-vorticity distribution on a solid body to the closure of the pressure distribution on the body, and by selecting an appropriate axis of reference for each general point concerned.

References

[1] Porthouse, D.T.C. (1983). *Numerical simulation of aerofoil and bluff body flows by vortex dynamics*. Ph.D. Thesis, University of Newcastle upon Tyne.

[2] Prandtl, L. and Tietjens, O.G., *Fundamentals of hydro- & aerodynamics.* United Engineering Trustees, Inc. 1934. Reprinted by Dover Publications, New York, 1934.