13th Australasian Fluid Mechanics Conference
Monash University, Melbourne, Australia
13-18 December 1998

The two dimensional, self-similar, compressible Taylor vortex

Karl von Ellenrieder and Brian J. Cantwell[†]

Department of Mechanical Engineering
Monash University, Clayton, VIC, Australia
[†]Department of Aeronautics & Astronautics
Stanford University, Stanford, CA, USA

ABSTRACT

When the compressibility of the fluid within a simple two dimensional vortex is important, an analysis of the flow must account for the effects of heat conduction, viscous dissipation, compressibility, and radial convection on the motion and structure of the vortex. The purpose of this paper is to examine these effects for a two dimensional Taylor vortex. Exact and numerical similarity solutions for the compressible perturbations to an incompressible, two dimensional, axisymmetric Taylor vortex are presented. The solutions allow one to explicitly determine the decay rates of the velocities and thermodynamic variables in the flow. An examination of the solutions also reveals that the temperature, density, entropy, and radial velocity in the vortex are strongly dependent on Prandtl number.

INTRODUCTION

A compressible flow is one in which a change in pressure over a characteristic lengthscale of the flow results in a corresponding change in density. Similarly, a compressible vortex contains large tangential velocities which create strong pressure gradients; these pressure gradients, in turn, produce substantial density variations across the vortex.

For example, in the two dimensional compressible vortex experimentally studied by Mandella (1987) the maximum tangential velocity in the vortex is about 230 m s^{-1}. This gives a vortex Mach number of $M \approx 0.67$, based on the far-field sound speed. The radial profile of the density measurements is nearly Lorentzian in shape, and monotonically decreases from 1.79 kg m^{-3} in the far-field to 0.80 kg m^{-3} at the vortex centre — a 55% drop in density. The corresponding pressure profile is also approximately Lorentzian and decreases from a maximum of $152\,100 \text{ N m}^{-2}$ in the far-field to about $44\,800 \text{ N m}^{-2}$ at the centre.

Compressible vortices are a significant feature of many fluid flows of physical and technological interest: turbulent combustion in engines, strong wing-tip vortices, the generation of aeroacoustic noise by jet aircraft, shock diffraction around sharp corners, and astrophysical flows such as solar coronae. Additionally, compressible vortex models have become popular for testing non-reflecting boundary conditions and as a basic component of numerical simulations for studying sound generation by shock/turbulence interaction. Yet, despite the frequent occurrence of compressible vortex flows in nature, the current number of analytical descriptions of this type of flow is quite small (Colonius, Lele, & Moin 1991; Mandella 1987). In light of this, we will examine analytical self-similar solutions for a slightly-compressible Taylor vortex (Taylor 1918) to gain insight on the fundamental physical behaviour and structure of compressible vortices. The vortex is a two-dimensional, axisymmetric, viscously-decaying, free vortex; the far-field flow conditions and the coefficients of viscosity and thermal conductivity are assumed to be constant.

Approach

The full Navier-Stokes equations will not yield similarity solutions for a free compressible, viscous vortex with constant far-field flow conditions. However, similarity solutions can be found for the simplified set of equations developed by Colonius *et al.* (1991). This equation set describes the evolution of the compressible perturbations to a incompressible, viscously-decaying, reference flow. It is assumed that the flow does not contain acoustic waves and that the viscosity μ and thermal conductivity κ of the fluid are constant. Self-similar solutions to the weakly-compressible equations can be found by applying a three parameter Lie group transformation (von Ellenrieder 1998). Here, the details of the solution method are omitted for brevity, and we focus on discussing the solutions when the reference flow is a two dimensional, incompressible, Taylor vortex.

SOLUTIONS

The tangential and radial velocities (v and u, respectively) and the thermodynamic variables (pressure p, density ρ, and temperature T) are scaled as shown in equation (1). v_m is a reference velocity (the maximum tangential velocity in the vortex's initial velocity profile) and l_i is the radial location of v_m. Far-field flow quantities are denoted with a subscript ∞, r is the radial coordinate, and t is time. The term $\tilde{p}$ is similar to the standard definition of the dynamic pressure coefficient in aerodynamics and represents the normalized deviation of the local pressure p from the far-field pressure p_∞.

$$\left.\begin{aligned} &\tilde{r} = \frac{r}{l_i}, \quad \tilde{t} = \frac{v_m t}{l_i}, \quad \tilde{v} = \frac{v}{v_m}, \\ &\tilde{u} = \frac{u}{v_m}, \quad \tilde{\rho} = \frac{\rho}{\rho_\infty}, \quad \tilde{p} = \frac{p - p_\infty}{\rho_\infty v_m^2}, \\ &\tilde{T} = \frac{T}{T_\infty}, \quad M = \frac{v_m}{a_\infty}, \quad \tau = \frac{\tilde{t}}{\tilde{Re}}, \\ &\tilde{Re} = \frac{\rho_\infty v_m l_i}{\mu}, \quad Pr = \frac{\mu C_p}{\kappa} \end{aligned}\right\} \tag{1}$$

The specific heat at constant pressure is denoted as C_p and is assumed to be constant.

Each of the dependent flow variables is approximated as

$$\tilde{f} = \tilde{f}_0 + M^2 \tilde{f}_1 + O(M^4), \tag{2}$$

where $\tilde{f}$ may represent any dependent flow variable. Terms of $O(1)$ are collected to yield equations describing an incompressible reference flow, and $O(M^2)$ terms are collected to give expressions for the compressible perturbations to the reference flow. The reference flow terms are designated with a subscript 0 and the $O(M^2)$ perturbations are denoted by the subscript 1.

The solution for the Taylor vortex reference flow (figure 1) is given by,

$$\tilde{\omega}_0 = \frac{C_1(1-\eta)e^{-\eta}}{(\tau + \tau_i)^2}, \tag{3a}$$

$$\tilde{v}_0 = \frac{C_1 \eta^{1/2} e^{-\eta}}{(\tau + \tau_i)^{3/2}}, \tag{3b}$$

$$\tilde{p}_0 = -\frac{C_1^2 e^{-2\eta}}{4(\tau + \tau_i)^3}, \tag{3c}$$

where $\eta = \tilde{r}^2/4(\tau + \tau_i)$. Let $C_1 = exp(1/2)/2$ and $\tau_i = 1/2$ so that the maximum tangential velocity is $\tilde{v}_0 = 1$ at $\tilde{r} = 1$ when $\tau = 0$. The reference flow temperature and density are taken to be constant, and are normalized to $\tilde{T}_0 = \tilde{\rho}_0 = 1$.

As the vortex decays, the total angular momentum of the reference flow is constant. The tangential velocity increases linearly near the vortex centre, and then decreases as $\sim \exp(-\tilde{r}^2)$ in the far-field. The vorticity in the centre of the vortex is positive, and changes sign near the core radius, $\tilde{r} = 1$ — there is zero total circulation in the reference flow. The pressure is lowest in the centre of the vortex, and increases monotonically with $\tilde{r}$.

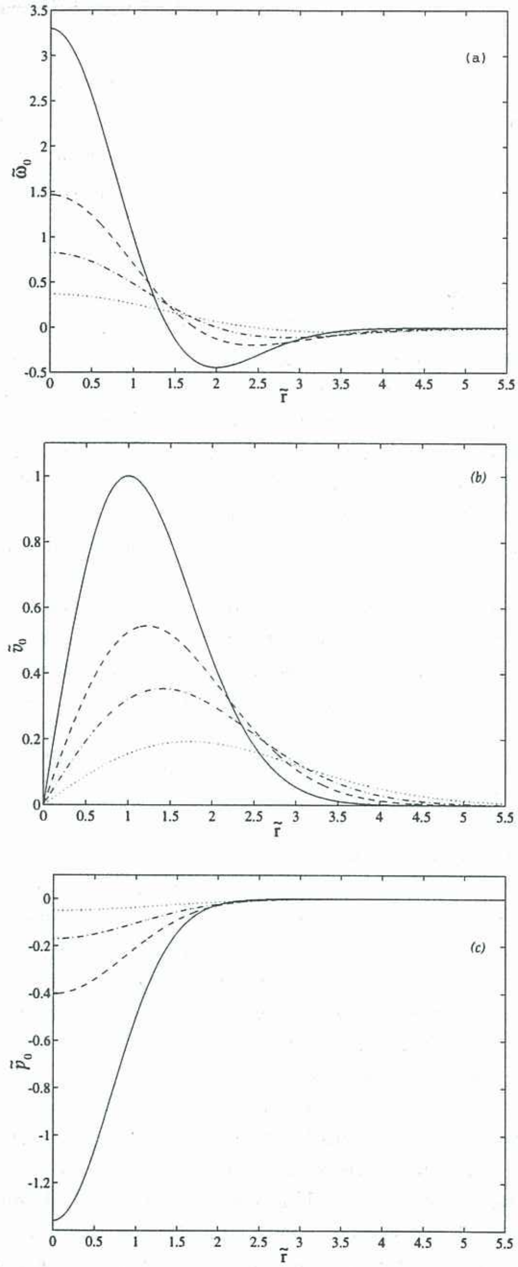

Figure 1: Taylor vortex reference flow (a) vorticity, (b) tangential velocity, and (c) pressure. Four different times are shown: ——, $\tau = 0$; – – – –, $\tau = 1/4$; —·—, $\tau = 1/2$, and ······, $\tau = 1$.

The self-similar forms of the solutions for the compressible perturbations when $Pr = 1$ are

$$\left.\begin{aligned} \sigma(\eta) &= -\frac{C_1^2(\gamma-1)}{4}e^{-2\eta}(2\eta+1) \\ &\quad +\frac{B_1}{2}e^{-\eta}\left(\eta^2-4\eta+2\right), \end{aligned}\right\} \quad (4a)$$

$$\left.\begin{aligned} \beta(\eta) &= \frac{C_1^2}{4}e^{-2\eta}\left[2\eta(\gamma-1)-1\right] \\ &\quad -\frac{B_1}{2}e^{-\eta}(\eta^2-4\eta+2), \end{aligned}\right\} \quad (4b)$$

$$\left.\begin{aligned} \zeta(\eta) &= -\frac{C_1^2(\gamma-1)}{2}\eta e^{-2\eta} \\ &\quad +\frac{B_1}{2}e^{-\eta}(\eta^2-4\eta+2), \end{aligned}\right\} \quad (4c)$$

and

$$\left.\begin{aligned} \alpha(\eta) &= \frac{C_1^2 e^{-2\eta}}{4\eta^{\frac{1}{2}}}\left[(2-\gamma)\right. \\ &\quad \left.+\eta(1-2\gamma)+2\eta^2(\gamma-1)\right] \\ &\quad -\frac{B_1\eta^{\frac{1}{2}}e^{-\eta}}{2}\left[\eta^2-6\eta+6\right] \\ &\quad +\frac{G_1}{\eta^{\frac{1}{2}}}. \end{aligned}\right\} \quad (4d)$$

We use the data of Mandella (1987) to pick physically realistic values of the integration constants: $G_1 = -C_1^2(2-\gamma)/4$, so the radial velocity is zero at $\tilde{r} = 0$; $B_1 = -1.67 \times 10^{-2}$ giving a 55% drop in density from the far-field to the vortex centre when $M = 0.67$ and the ratio of specific heats $\gamma = 1.4$.

The physical form of the flow variables is recovered with the relations:

$$\left.\begin{aligned} \sigma(\eta) &= \tilde{T}_1(\tau+\tau_i)^3, \\ \beta(\eta) &= \tilde{\rho}_1(\tau+\tau_i)^3, \\ \zeta(\eta) &= \tilde{s}_1(\tau+\tau_i)^3, \\ \alpha(\eta) &= \tilde{u}_1^*(\tau+\tau_i)^{7/2}, \end{aligned}\right\} \quad (5a\text{–}e)$$

where $\tilde{u}_1^* = \tilde{u}_1\tilde{Re}$. Equations ($5a$–$e$) allow one determine the decay rates of each of the flow quantities by simple inspection

$$\tilde{T}_1, \tilde{\rho}_1, \tilde{s}_1 \propto \tau^{-3}, \quad \tilde{u}_1^* \propto \tau^{-7/2}.$$

Prandtl number dependence

We will use values of $Pr = 0.5$, 0.72, and 1.0 to get a feel for the behavior of the $O(M^2)$ solutions when Pr is varied over an extreme range.[1]. The solutions for $Pr \neq 1$ are obtained using series solutions and numerical integration (von Ellenrieder 1998).

Consider how the $O(M^2)$ solutions vary with the Prandtl number (figure 2). Pr represents the ratio of viscous to thermal diffusion. When $Pr = 1$, heat and viscosity diffuse at the same rate; when $Pr < 1$, heat diffuses faster. Therefore, when the temperature profiles for different values of Pr are plotted on the same graph, and each profile is set to the same temperature at $\tilde{r} = 0$ and $\tau = 0$, the temperature distribution for the curve with the smaller value of Pr will extend further from the origin (figure 2a).

The variation of the radial velocity perturbation with Pr is shown in figure 3. As $\tilde{r} \to \infty$ the compressible perturbation terms vanish and the far-field flow is essentially incompressible. Because of this, one would expect the asymptotic behavior of the radial velocity around the vortex to have the same form as the radial velocity around a point mass sink in a two dimensional, incompressible flow. As figure 3 shows, the far-field radial velocity has this property: it varies with $\tilde{r}$ as $\tilde{u}_1^* \sim 1/\tilde{r}$.

The far-field behavior of the radial velocity is the same for all values of Pr. This is because slightly-compressible vortices are always compressed (Colonius *et al.* 1991; von Ellenrieder 1998). In figure 3 we see that for the Taylor vortex $\tilde{u}_1^*$ is negative for all values of $\tilde{r}$. The magnitude of $\tilde{u}_1^*$ is zero at $\tilde{r} = 0$, increases almost linearly to a maximum, and in the far-field decreases like $1/\tilde{r}$. A comparison of figure 2b and figure 3 reveals that the magnitude of the radial velocity is larger in those regions of the flow that are less compressed.

CONCLUDING REMARKS

The above analysis describes some of the basic physical properties of a self-similar, slightly-compressible Taylor vortex and is a start to understanding compressible vortices in general. Also, the use of the $Pr = 1$ closed form Taylor vortex solutions can be a quick and simple means of validating and initializing numerical simulations of flows containing free, two-dimensional, compressible vortices.

Here, we have assumed the viscous diffusivity and thermal conductivity of the fluid are constant. In reality both of these quantities are temperature dependent. An interesting extension of the present work would be to include the temperature dependence of μ and κ by using power law models.

A full characterization of the flow during the formation and subsequent evolution of the compress-

[1] The Pr for most monatomic and diatomic gases at atmospheric pressure lies within the range $0.67 \leq Pr \leq 0.85$ for temperatures between $100\text{K} \leq T \leq 1300\text{K}$. For a given gas, the Pr is roughly constant, even at large temperatures. The theoretical value of Pr for a monatomic gas is $Pr = 2/3$

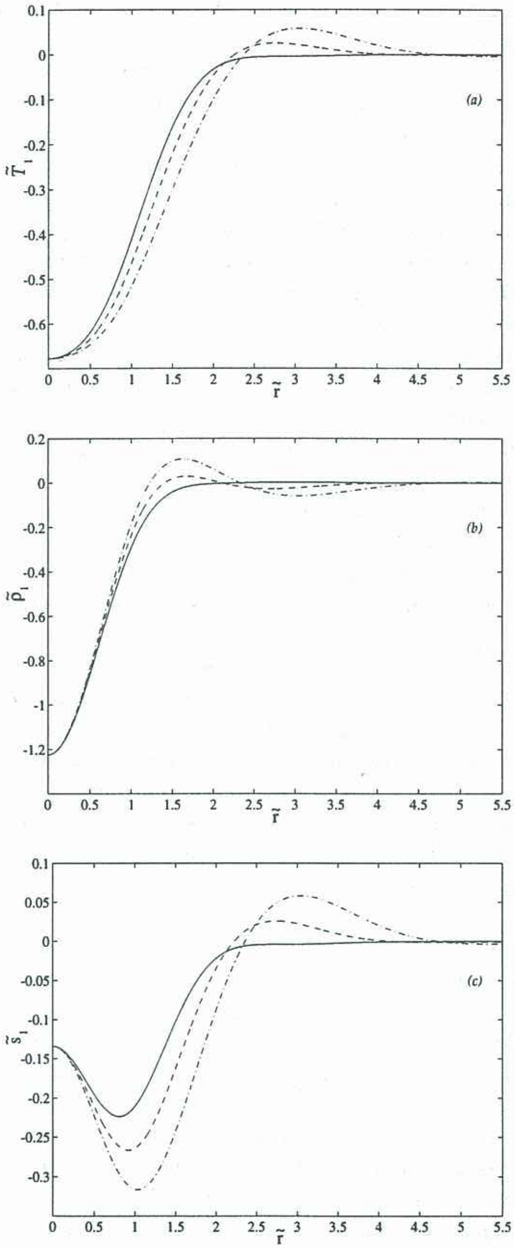

Figure 2: Pr dependence of the series solution when $\tau = 0$. Radial distributions of $O(M^2)$ (a) temperature, (b) density, and (c) entropy, and are shown for three different values of Pr:
——, $Pr = 1.00$; ----, $Pr = 0.72$; and —·—, $Pr = 0.50$.

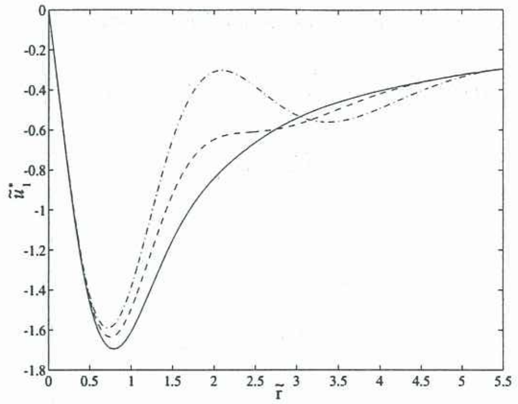

Figure 3: Pr dependence of the solution for a Taylor reference flow when $\tau = 0$. Radial distribution of the $O(M^2)$ radial velocity is shown for three different values of Pr:
——, $Pr = 1.00$; ----, $Pr = 0.72$; and —·—, $Pr = 0.50$.

ible vortex studied by Mandella (1987) has still not been done. Since compressible vortices are present in many flows of engineering and physical interest, a thorough knowledge of the formation process and structure of a single compressible free vortex is of fundamental importance. Therefore, further computational and experimental effort should be directed towards more fully understanding the flow studied by Mandella (1987).

ACKNOWLEDGEMENT

This work was supported, in part, by the NASA Graduate Student Researchers Program and the NASA Ames-Stanford Joint Institute for Aeronautics and Acoustics NCC 2-55.

REFERENCES

Colonius, T., Lele, S. K. & Moin, P., "The free compressible viscous vortex", *J. Fluid Mechanics*, 230:45–73, 1991.

Mandella, M. J., "Experimental and analytical studies of compressible vortices", Ph.D. thesis, Dept. of Applied Physics, Stanford University, 1987.

Taylor, G. I., "On the dissipation of eddies", 1918. Reprinted in *The Scientific Papers of Sir Geoffrey Ingram Taylor. Vol II. Mechanics of Fluids: Misc. Papers.* (ed. G. K. Batchelor). Cambridge University Press, 1971.

von Ellenrieder, K., "Self-similar compressible free vortices", Ph.D. thesis, Dept. of Aeronautics & Astronautics, Stanford University, 1998. (Also available as NASA JIAA report TR 121.)