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THE EFFECT OF BOTTOM FRICTION ON INSTABILITY AND WAVE OVER-REFLECTION IN SHALLOW WATER

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ABSTRACT

In a two-dimensional shear flow of shallow water, the bottom friction relates uniquely the spanwise profile of the undisturbed depth-averaged velocity to the bottom topography. If the basic flow varies weakly in the spanwise direction, the local analysis of stability at every spanwise position gives the region of the flow parameters, for which the classic hydraulic instability due to the bottom friction cannot occur. In this region, the linear analyses of the waves scattering and instability due to the lateral shear can be performed effectively by means of the frictionless linearized equations if both the bottom slope and friction are equally small.

In the absence of hydraulic instability, the waves can be amplified only near the critical layers, where their streamwise phase velocity equals to the velocity of the basic flow. Two physical mechanisms of this amplification exist. The first one is similar to that suggested by Takehiro and Hayashi (1992) for a linear frictionless shallow water flow. The incident and transmitted waves carry energy of opposite signs, which results into increasing of the amplitude of the reflected wave compared to that of the incident one. This mechanism of over-reflection operates for any combination of the flow parameters. The other mechanism is similar to Landau damping in plasma flows; it is related to the energy exchange between the waves and fluid particles at the critical layers due to the velocity synchronism. It may lead to either additional amplification or damping of the waves for different flow conditions. In particular, its significance can be reduced by stronger bottom friction.

If the basic flow has uniform potential vorticity, Landau damping is negligible. Furthermore, over-reflection occurs then without any exchange of energy between the waves and induced mean flow. If the proper feed-back is provided by another critical layer, the net over-reflection results into the formation of self-exited trapped modes, which can be interpreted as the global instability of a hypothetical one-dimensional state that is spatially developing and has no region of local instability.

INTRODUCTION

The shallow-water approximation is based on the assumption that the horizontal lengthscale of the flow (*e.g.* the typical wavelength) is much larger than the typical depth. It provides a useful reduction of dimensions in many hydraulic and geophysical applications. Thus, the vertical direction is eliminated from the consideration by averaging the equations of motion over the flow depth. The shear stresses in the horizontal plane result then into the friction at the bottom. The instability that is caused by the friction is often referred to as hydraulic; its most recognized consequence is the formation of roll waves in steep open channels and other hydraulic structures.

If the depth-averaged basic flow varies in the spanwise direction, the other type of instability can occur (Satomura, 1981; Chu, Wu and Khayat, 1991; Takehiro and Hayashi, 1992; Knessl and Keller, 1995; Shrira *et al.*, 1997). The instability mechanism is based on the wave amplification near the critical layers, where the streamwise phase speed of the waves equals to the local streamwise velocity of the basic flow. In a wave scattering problem, the amplification appears in the form of over-reflection—the amplitude of the wave reflected from the critical layer is higher than that of the incident one. Strictly speaking, the viscous analysis is required near the critical layer. However, the instability can be efficiently predicted by pure inviscid theory if one treats properly the singularity that may occur in the inviscid equations (Grimshaw, 1980).

For a linear shear flow of shallow water along a frictionless horizontal plane, Knessl and Keller (1995, hereafter KK95) obtained an exact analytic solution for small disturbances and demonstrated that over-reflection occurs for any combination of the flow parameters. In the present paper, we extend their result to a more realistic flow of shallow water, for which the bottom slope and friction are not negligible. We utilize the combination of the comparison equation technique with the matching of the WKBJ approximation in the complex plane, which has been suggested by Basovich and Tsimring (1984, hereafter BT84).

FORMULATION

The basic flow is a free surface stream of an incompressible fluid (Fig.1). The depth of the basic flow $H(y)$ is measured normally to the free surface. The vertical variation of the flow parameters is assumed to be negligible, and it does not affect the depth-averaged equations. The depth-averaged velocity of the basic flow has only one (streamwise) non-zero component $U(y) > 0$.

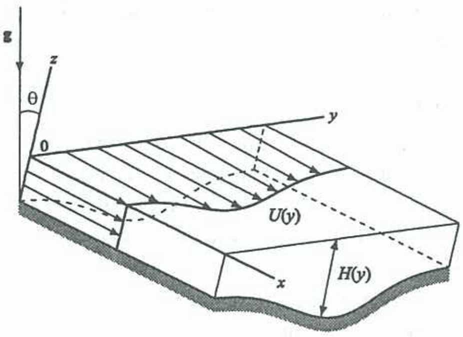

Figure 1: Flow geometry.

The governing equations are two-dimensional Saint Venant shallow-water equations. The bottom friction is accounted via the term $\tau = \lambda \mathbf{u} f(|\mathbf{u}|)\, h^{-1}$, where λ is a non-dimensional friction factor, $\mathbf{u}$ is the depth-averaged velocity vector, and $f(|\mathbf{u}|)$ is an arbitrary function such that $f > 0$ and $f' > 0$. This form generalizes some of *ad hoc* formulas that are often used in applications (for references, see Yakubenko and Shugai, 1998).

The flow depth is presented as

$$H(y) + \hat{h}(y) \exp(ik_x x - i\omega t),$$

and a similar decomposition is performed for the velocity. Then, the equations of motion yield for the basic flow

$$F^{-2} \tan(\theta) - \tau(H, U) = 0, \qquad (1)$$

in which $F = U_{typ}\left[H_{typ} g \cos(\theta)\right]^{-1/2}$ is the Froude number.

LIMITS OF THE APPROACH

For $U \equiv H \equiv 1$, one can take $\hat{h}(y) = \exp(ik_y y)$, and the linearized equations of motion lead to the dispersion relation. From the relation and (1), one can shown that the flow is temporally unstable only if

$$\tan(\theta) > \lambda \left[f(1) + f'(1)\right]^2 f^{-1}(1), \qquad (2)$$

which is independent of k_y. Furthermore, in the case of instability, the dispersion relation reveals that disturbances of infinitely short wavelength are the most amplified. It may be argued that this fact contradicts the long-wave nature of the shallow water approximation. The problem is resolved if the internal lateral friction is taken into account in addition to the bottom friction (Yakubenko and Shugai, 1998). However, this requires other terms of higher order to be retained. To avoid this complication, we assume in the remainder that (2) is not satisfied.

The other important limiting case is the flow in a horizontal frictionless channel ($\theta = \lambda = 0$). Then, the following equation for $\hat{h}(y)$ is obtained:

$$\hat{h}'' - \partial_y \left[\log\left(\Lambda^2\right)\right] \hat{h}' + k_x^2 \left(F^2 \Lambda^2 - 1\right) \hat{h} = 0, \qquad (3)$$

in which $c = \omega / k_x$ is the streamwise phase velocity, and $\Lambda^2 = (U - c)^2 H^{-1}$. Equation (3) is similar to those for shear flows of stratified fluids and of compressible gases (BT84; Lindzen, 1988; Drazin and Davey, 1977, and references therein). Since (1) disappears, the depth and velocity of the basic flow can vary independently.

Equation (3) is further simplified if the bottom is uniform. In shallow water, this case was investigated first by Satomura (1981). For $H \equiv 1$ and $U(y) \equiv y$, Takehiro and Hayashi (1992, hereafter TH92) have investigated numerically how a wave packet interacts with the critical layer, and KK95 have obtained an exact solution of the wave scattering problem.

It must be noted that even in the frictionless case, the velocity profile can be related uniquely to the bottom topography if the entire flow is subjected to rotation. The effect may be significant in many geophysical applications (see McPhaden and Ripa, 1990 for a review).

WEAK BOTTOM FRICTION

We assume that both the bottom friction and slope are equally small in the following sense:

$$\lambda \ll 1, \quad \gamma^2 = \lambda \cot(\theta) = O(1). \qquad (4)$$

The resulting equations for the basic flow and disturbances are the following:

$$H = \gamma^2 F^2 U f(U), \qquad (5)$$

$$\hat{h}'' - \partial_y \left[\log\left(\Lambda^2\right)\right] \hat{h}' + k_x^2 \left(\gamma^{-2} \Lambda^2 - 1\right) \hat{h} = 0, \qquad (6)$$

in which

$$\Lambda^2(y) = [U(y) - c]^2 \{U(y) f [U(y)]\}^{-1}. \quad (7)$$

Thus, in the leading order approximation, the problem is similar to that of the frictionless case. However, the depth and velocity of the basic flow cannot vary independently any more.

Equations (2) and (4) give that hydraulic instability cannot occur if

$$\gamma^2 > [f(1) + f'(1)]^2 f^{-1}(1). \quad (8)$$

The other important restriction concerns the critical layers $y = y_c$, where the velocity of the basic flow equals the phase speed of the waves. Equation (6) has a singularity at $y = y_c$. As a result, its solution has there a branch point. (Most general treatment of the problem and an overview of related works were presented by Grimshaw, 1980.) Therefore, the problem of choice of the proper branch arises. However, the choice can be made easily if one keep in mind that (6) is obtained by neglecting the friction terms. Thus, the actual position of the singularity is given by

$$y = y_c + i\left\{\frac{\partial_u \tau(H,U)}{k_x U'}\right\}_{y=y_c} + O(\lambda^2), \quad (9)$$

where $\partial_u \tau(H,U) = O(\lambda)$. Hence, the solutions of the reduced equation (6) at the real y-axis can be matched around y_c in the complex y-plane. Furthermore, since $\partial_u \tau > 0$, equation (9) shows that the bypass must be taken in the lower complex half-plane for $k_x U'(y_c) > 0$, and in the upper one for $k_x U'(y_c) < 0$.

SCATTERING PROBLEM

The basic flow velocity is given by a monotonous function $U(y)$ such that $U(y \to \pm\infty) \to U_\pm = \text{const}$. The corresponding bottom topography is given by $H = \gamma^2 F^2 U^2$. If the incident wave may come from either direction ($y = \pm\infty$), the assumption $U'(y) > 0$ entails no loss of generality. We assume further that a critical layer occurs in the flow, and that the basic flow varies weakly at the typical wavelength. This weak variation is made explicit by introducing $\tilde{U}(Y)$, $\tilde{\Lambda}(Y)$, and $\tilde{h}(Y) \equiv \hat{h}(\varepsilon^{-1}Y)\tilde{\Lambda}^{-1}(Y)$, in which $Y = \varepsilon(y - y_c)$ and $\varepsilon \ll 1$. Equation (6) takes then the standard form

$$\tilde{h}'' + P(Y)\tilde{h} = 0, \quad (10)$$

in which $P \equiv \varepsilon^{-2}k_Y^2 + \tilde{\Lambda}^{-1}\tilde{\Lambda}'' - 2\tilde{\Lambda}^{-2}\left(\tilde{\Lambda}'\right)^2$, and

$$k_Y^2 = k_x^2\left(\gamma^{-2}\tilde{\Lambda}^2 - 1\right). \quad (11)$$

The scattering potential $P(Y)$ has a second order pole at $Y = 0$ and two zeroes Y_1 and Y_2. Following BT84, we approximate it near the origin by a standard potential that provides the following general solution in terms of Whittaker's functions:

$$k_Y^{-1/2}(Y)\left[C_1 W_{\kappa,3/2}(S) + C_2 W_{-\kappa,3/2}(-S)\right], \quad (12)$$

in which C_1 and C_2 are arbitrary constants,

$$\kappa = \frac{1}{2} i\varepsilon\tilde{\Lambda}''(0)\left[k_Y(0)\tilde{\Lambda}'(0)\right]^{-1},$$

and $S(Y) \equiv 2i\varepsilon^{-1}\int_0^Y k_Y(\xi)\,d\xi$.

The solution (12) breaks down near the points Y_1 and Y_2. If the WKBJ approximation is used, these are simple turning points, and the standard WKBJ matching in the complex plane can be utilized. This implies an accurate choice of the branches of $k_Y(Y)$, since a scattering problem requires precise identification of the incident and scattered waves. Thus, each branch must be associated with a wave, for which the direction of propagation must be determined uniquely for the entire real Y axis. Since the problem must be treated for $\mathrm{Im}(\omega) = 0$ as a limiting case of that for $\mathrm{Im}(\omega) > 0$, a proper direction of matching can be established unambiguously for each turning point.

If $k_x > 0$, one has $k_x\tilde{U}'(0) > 0$, so that the pole $Y = 0$ must be bypassed in the lower complex half-plane. This case is mathematically similar to the problem formulated by BT84. However, they matched the singularity in the incorrect (upper) half-plane (BT84, Fig.6, p. 245) and used incorrect values of the Stokes multipliers. Therefore, we have revised their matching procedure. The following reflection and transmission coefficients are obtained:

$$|R_m|^2 \simeq 1 + e^{-2d} + (-1)^m 2\pi\kappa e^{-2d_m}, \quad (13)$$

$$|T|^2 \simeq e^{-2d}\left[1 - \pi\kappa e^{-2d_1} + \pi\kappa e^{-2d_2}\right], \quad (14)$$

in which $m = 1$ and 2 must be taken for the left- and right-scattering, respectively, $d_{1,2} = \varepsilon^{-1}\left|\int_0^{Y_{1,2}} k_Y(Y)\,dY\right| \gg 1$, and $d = d_1 + d_2$. Equations (13) and (14) yield

$$|R_m|^2 - |T|^2 \simeq 1 + (-1)^m 2\pi\kappa e^{-2d_m},$$

which generalizes the result of KK95.

For $k_x < 0$, one has $k_x\tilde{U}'(0) < 0$, so that the bypass must be taken in the upper complex half-plane. It can be shown that the formulas for the reflection and transmission coefficients can be obtained from (13) and (14) simply by swapping first d_1 and d_2, and then R_1 and R_2.

OVER-REFLECTION vs LANDAU DAMPING

The potential vorticity of the basic flow and waves are given $Q = -U'H^{-1}$ and $q =$

$(\partial_x v - \partial_y u - Qh)\, H^{-1}$, respectively. If the potential vorticity of the basic flow is uniform ($Q' \equiv 0$), one can show that the potential vorticity of the waves q is conserved. Two important cases can be distinguished. In the first one, q is zero initially. Then, it is zero at any moment of time (which excludes vorticity waves but retains surface-gravity waves). This case has been discussed in details by TH92. In particular, they have shown that over-reflection occurs without energy (and momentum) exchange between the waves and induced mean flow (the energy flux is continuous across the critical layer). The excitation of the transmitted wave of negative energy increases the energy of the reflected wave compared to that of the incident one, which in its turn results into increasing of the wave amplitude, *i.e.* into over-reflection. In equation (13), it is accounted by the term e^{-2d} that is always positive. Furthermore, this mechanism of over-reflection operates as well for non-zero Q' and q, since (13) does not relay on any assumptions about them. Thus, over-reflection can be explained as an interaction between two waves carrying energy of opposite signs (Acheson, 1976; Craik 1985; Stepanyants and Fabrikant, 1988; Shrira *et al.*, 1997).

Possible direct energy exchange between the waves and fluid particles due to the velocity synchronism near the critical layer is often referred to as Landau damping because of its similarity to that suggested by Landau in plasma physics (for references, see Briggs *et al.*, 1970 and Stepanyants and Fabrikant, 1989). Physically, it is related to the trapping of fluid particles (originally, electrons) whose velocities are close to the wave phase speed between adjacent peaks. It must be noted, however, that the actual trapping process cannot be accounted completely by the linear theory (for further discussion, see Briggs *et al.*, 1970 and Craik, 1985). The waves interact efficiently with those particles that have velocities from the interval $(c-\delta, c+\delta)$, in which $0 < \delta \ll 1$. The difference between the number of particles (per unit length in the x-direction) that propagate faster and slower than the wave can be expressed as

$$N \simeq \delta^2 Q'(y_c) \left[U'(y_c)\, Q^2(y_c)\right]^{-1}, \qquad (15)$$

in which only the leading term is retained. Because of the assumption $U'(y_c) > 0$, the sign of N coincides with that of the gradient of the basic potential vorticity.

If $N > 0$, the waves altogether receive energy due to the interaction. (This extra energy is compensated by the corresponding alterations of the energy of the induced mean flow.) The opposite case is more spectacular, since the waves carrying negative energy can be amplified by transferring energy to the mean flow. The reflection and transmission coefficients can be easily rewritten in terms of N, since

$$\kappa = \frac{1}{2} k_x^{-1} \delta^{-2} U'(y_c)\, Q(y_c)\, N.$$

To illustrate the effect of the bottom friction, we consider the case $f(U) \equiv U$; the corresponding law of the bottom friction is called the Chezy formula, it has been used in hydraulic practice for more than a century. The instability condition (2) takes the celebrated form $\tan(\theta) > 4\lambda$. Thus, for weak friction, the hydraulic instability is absent if $\gamma > 1/2$. In Fig. 3, the reflection coefficient is shown for the following velocity profile:

$$U(y) = \frac{1}{2}\left[U_+ + U_- + (U_+ - U_-)\tanh(\varepsilon y)\right].$$

Thus, larger values of γ, and thus stronger bottom friction, suppress Landau damping.

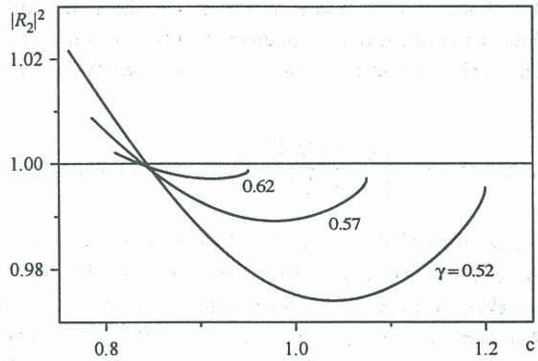

Figure 2: Right-reflection coefficient for tanh-profile of the basic velocity ($U_- = 0.5$, $U_+ = 2.5$, $\varepsilon = 0.03$, and $k_x = 0.06$).

TRAPPED MODES AND GLOBAL INSTABILITY

The feed-back for the over-reflected wave can be provided by either a lateral boundary of the flow or another critical layer. The net reflection can lead then to the formation of the so-called trapped modes that are eigenfunctions of equation (10) together with the radiation conditions for $y \to \pm\infty$. For example, two simple but important particular cases are a submerged trough and ridge. In both cases, the scattering potential $P(Y)$ can have two critical layers and four turning points.

The eigenfrequencies are found from the quantization relation. It can be shown that in the case of over-reflection, some of these frequencies have positive imaginary part, thus providing instability (Acheson, 1976; Lindzen, 1988; TH92; KK95). In the theory of stability of parallel flows, the quantization relation is considered as the dispersion relation between the wavenumber k_x and frequency ω. Alternatively, k_x can be treated merely as a parameter, and the stability problem is considered for a fictitious flow that is one-dimensional and spatially developing. Then, $k_Y(\omega, Y; k_x)$ presents the local wave number and the local dispersion relation is given by (11). In this approach, the eigenfrequencies ω and the corresponding eigenfunctions $\tilde{h}(Y;\omega)$ are called global frequencies and global modes, respectively (more details can be found in Huerre and Monkewitz, 1990 and Yakubenko, 1997).

In recent years, the problem of stability of spatially developing flows has attracted much attention in fluid dynamics. One of the central questions in the studies was how the local properties of the flow are related to its global stability. In particular, Brevdo and Bridges (1997) have demonstrated by means of a mathematical example that the presence of a region of local instability is not necessary for the global instability. In the mater of fact, the same conclusion is hidden in a huge number of the existing stability studies. For example, dispersion relation (11) indicates no local instability. (Moreover, we have assumed the region of the flow parameters, for which the local hydraulic instability would never occur even if we retained the terms due to the bottom friction.) Nevertheless, some of the frequencies may have positive imaginary parts. As a conclusion, local instability is not necessary, at least formally, for the global instability of a spatially developing state. To our knowledge, though, a question whether over-reflection can occur in a real one dimensional flow remains open.

Finally, it must be noted that the linear theory presented has limitations, and the wave transformation near the critical layer may involve complex non-linear phenomena (further discussion can be found in Craik, 1985; Grimshaw, 1994; Shrira *et al.*, 1997).

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