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# NUMERICAL MAGNETOHYDRODYNAMIC MODELLING OF A CONDUCTING FLUID IN A RECTANGULAR DUCT

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## ABSTRACT

The flow of a conducting fluid under the influence of an imposed electric and magnetic field is analysed using Computational Fluid Dynamic techniques. A finite volume method is used to solve the Navier-Stokes equations for the fluid flow variables. The Lorentz force is obtained by solving Maxwell's equations via the same finite volume technique. Velocity profiles for a two dimensional flow are predicted for a number of different Hartmann numbers and compare well to the analytic solution. The more complex case of three dimensional flow is also modelled and demonstrates that the model is able to predict the Hartmann layer and high velocity regions which occur near walls.

## NOMENCLATURE

| | |
|---|---|
| $a$ | characteristic length |
| $\mathbf{A}$ | magnetic vector potential |
| $\mathbf{B}$ | magnetic field |
| $\mathbf{E}$ | electric field |
| $\mathbf{F}_L$ | Lorentz force |
| $\mathbf{J}$ | current density |
| $M$ | Hartmann number |
| $p$ | pressure |
| $\mathbf{u}$ | velocity |
| $\phi$ | magnetic scalar potential |
| $\rho$ | density |
| $\sigma$ | conductivity |
| $\mu$ | dynamic viscosity |
| $\mu_o$ | magnetic permeability |

## INTRODUCTION

When a conducting fluid is moving relative to a magnetic field, an electric current is induced in the fluid. When the magnetic field and electric fields are perpendicular an electromagnetic force, known as the Lorentz force, acts on the fluid. The Lorentz force, depending on its relative magnitude, may alter the behaviour of the fluid. Magnetohydrodynamics is an important field of fluid dynamics, applications include the Hall-Heroult cell for aluminium reduction, electromagnetic braking in continuous casting operations, electromagnetic pumping, levitation of molten metals and liquid metal cooling in fusion reactors. The current work is part of a larger program aimed at implementing and validating a MHD model suitable for solving industrial type problems.

An important problem concerning a conducting fluid is laminar Poisson flow in a rectangular duct under the influence of transverse electric and magnetic fields. The problem is known as Hartmann flow and is one of the simplest cases of MHD flows. In addition to being a reasonably simple type of flow it also has important application in reactor cooling channels and is similar, in principle, to electromagnetic braking and pumping applications.

The interaction between the fluid and the fields induces a Lorentz force which opposes the fluid flow. The result is that the velocity profile tends to flatten from the ideal Poisson laminar flow profile. The simple nature of the problem allows an analytic solution to be obtained for a two dimensional case. In the three dimensional case conductivity of the side walls of the duct becomes important introducing high speed flows near the duct walls and current flows known as the Hartmann layer. This paper presents the basis for the MHD model and demonstrates that such a model can readily be implemented in a commercial CFD code. The model is used to predict velocity profiles in two and three dimensional ducts.

## NUMERICAL MODEL

Prediction of the behaviour of conducting fluids under the influence of a magnetic field or MHD requires solution of the Navier-Stokes equations for hydrodynamic variables such as pressure and velocities and Maxwell's equations for the electric and magnetic fields. The two sets of equations are coupled together through the Lorentz force.

### HYDRODYNAMIC MODEL

For a laminar fluid the Navier-Stokes equations are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad (1)$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \nabla . \mu \nabla \mathbf{u} + \mathbf{F}_L \qquad (2)$$

### ELECTROMAGNETIC MODEL

The electric and magnetic fields are governed by Maxwell's equations (Feynman, 1964):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (3)$$

$$\nabla \cdot \mathbf{J} = 0 \qquad (4)$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad (5)$$

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} \qquad (6)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (7)$$

which are coupled to the Navier-Stokes equations via the Lorentz force:

$$\mathbf{F}_L = \mathbf{J} \times \mathbf{B} \quad (8)$$

## NUMERICAL SOLUTION PROCEDURE

The commercial CFD code, CFX4, is used to solve the hydrodynamic equations on a collocate body fitted grid using a finite volume method. Details of the hydrodynamic solution procedure are given in AEA (1997). Solution of the electromagnetic equations is now discussed.

Solution of equations (3) and (6) could be performed directly to give the electric and magnetic fields, however this would require solution of at least six equations for a three dimensional case. Additional constraints to enforce equations (4) and (7) would also be required. In the current model a potential approach is adopted. As the divergence of **B** is zero then **B** can be represented by the curl of the magnetic vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (9)$$

which naturally enforces equation (7).

Substitution of equation (9) into Faraday's Law, equation (3) gives:

$$\nabla \times \left( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \quad (10)$$

since the curl of the term in brackets is zero, a magnetic scalar potential can be defined as:

$$-\nabla \phi = \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \quad (11)$$

Equations (9) and (11) can be substituted into equations (5) and (6) to give:

$$\nabla \times \nabla \times \mathbf{A} = \sigma \mu_o \left( -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \times (\nabla \times \mathbf{A}) \right) \quad (12)$$

Making use of the identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

and the Coulomb Gauge ( $\nabla \cdot \mathbf{A} = 0$ ) the equation for **A** is:

$$\frac{\partial \mathbf{A}}{\partial t} - \nabla^2 \mathbf{A} = \sigma \mu_o \left( -\nabla \phi + \mathbf{u} \times (\nabla \times \mathbf{A}) \right) \quad (13)$$

Taking the divergence of Ohm's Law (equation (5)), equating to zero to satisfy current continuity (equation (4)) and substituting in equations (9) and (11) gives the following Poisson equation for the magnetic scalar potential:

$$-\nabla^2 \phi = -\nabla \cdot (\mathbf{u} \times (\nabla \times \mathbf{A})) \quad (14)$$

In CFX4 (AEA, 1997) it is possible to solve convection-diffusion transport equations for additional scalar variables, $\Phi$:

$$\frac{\partial \rho \Phi}{\partial t} + \nabla \cdot \left( \rho \mathbf{u} \Phi - \Gamma_\Phi \nabla \Phi \right) = S_\Phi \quad (15)$$

By removing the transient and convection terms where appropriate in equation (15) and specifying the appropriate diffusion and source terms, equations (13) and (14) can be solved using CFX4 on the same grid as the flow variables.

## TRIAL PROBLEM

The test problem is the steady flow of a conducting liquid through a long rectangular duct as shown in Figure 1. The direction of fluid flow is in the x direction into the page. A magnetic field of strength $B_o$ is applied in the y-direction. For all cases the fluid properties and velocity are selected to give a Reynolds number just over 900. The duct half height, $a$, is 25mm and the velocity is 0.1m/s. The initial velocity distribution at the duct entry is taken as the parabolic laminar profile and the flow profile is allowed to develop along the duct due to the influence of the imposed fields.

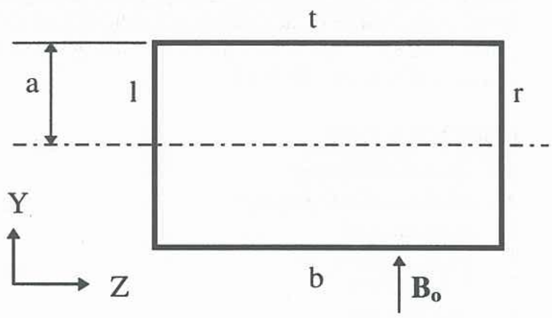


**Figure 1 :** Rectangular duct geometry

### Two Dimensional Case

The duct is assumed to be infinite in the z-direction with an imposed electric field acting in the z-direction. The top and bottom walls (t & b) have a non-slip boundary condition. Under these conditions the velocity will be uniform in the z-direction. If there is no Lorentz force then the velocity profile is dominated by viscous forces and can readily be shown to be parabolic with the centre-line velocity 1.5 time the mean velocity. When the magnetic and electric fields are applied a Lorentz force arises and opposes the fluid motion. Consequently viscous and Lorentz forces control the velocity distribution in the duct.

The Hartmann number is the ratio of Lorentz force to viscous force and defined as:

$$M = aB_o \sqrt{\frac{\sigma}{\mu}} \quad (16)$$

Clearly the simplest approach to model this problem is to observe that the Lorentz force is constant and add the required body force to the momentum equation in a two dimensional calculation as done by Hughes, *et al.* (1994). In this case a more complex solution procedure is adopted to test the solution procedure. A three dimensional grid with 50x50x5 cells in the x, y & z-directions is used with the electric field distribution obtained by solving equation (14). At the left and right walls (l & r) a free slip velocity condition is applied with a fixed potential and infinite conductivity. A grid sensitivity study indicated that the mesh was appropriate. The magnetic field is set to a constant value because when equation (9) is included the induced currents produce small secondary flows which are discussed in the next section.

Using this two dimensional flow approximation the momentum equations can be solved analytically. The solution for the velocity profile for an infinitely wide duct, as given by Sutton and Sherman (1965), is:

$$\frac{U}{U_o} = \frac{M\left[\cosh(M) - \cosh\left(M\, {Y}/{Y_o}\right)\right]}{M\cosh(M) - \sinh(M)} \quad (17)$$

The predicted velocity profile for a range of Hartmann numbers is shown in Figure 2. For the simulation the mass flow rate, Reynolds number and electric field strength remained fixed. Variation in the Hartmann number was achieved by altering the magnetic field strength. The discrete points are the model predictions while the continuous line is the solution of equation (17). Clearly the CFD model results agree very well with the analytic solution. At a Hartmann number of one half the profile is almost parabolic with a centre-line velocity of 1.46 only slightly lower than the 1.5 value if no magnetic field was applied. Increasing the magnetic field strength results in a progressive flattening of the velocity profile. For the $M = 25$ case the Lorentz force confines the viscous forces to the wall region only and the central core of the duct has a nearly uniform profile. To accurately capture the velocity profile in this near wall region a grid refinement study was required. The simplified case of a fixed Lorentz force was also run and produced identical results.

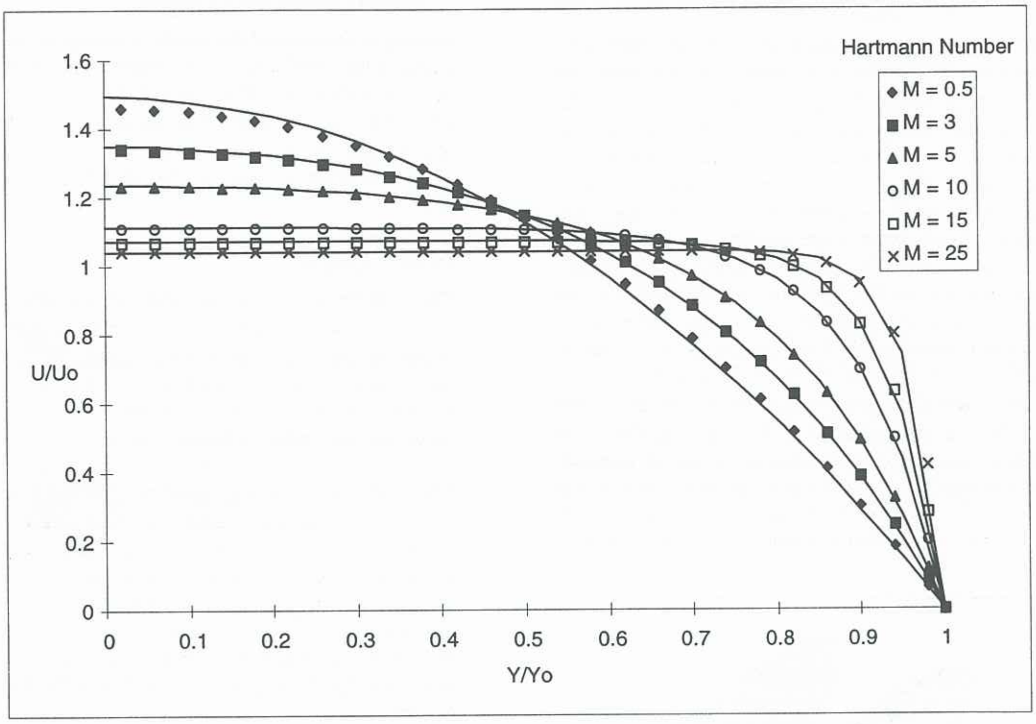


**Figure 2 :** Comparison of analytical (equation (17), lines) and CFD predicted velocity profiles (points) in a rectangular duct for various Hartmann numbers.

**Three Dimensional Case**

In equation (5) the $\mathbf{u} \times \mathbf{B}$ term is the induced current term. In the earlier two dimensional case this term is assumed small and ignored. For the case considered in this section the transverse external electric field of the previous section is removed. The u-velocity and the $B_y$ magnetic field components induce an electric potential in the z-direction. If the side walls of the duct are insulators then return current must be carried through thin fluid layers near the top and bottom walls of the duct known as Hartmann layers. When all duct walls are insulators the current flow through the core is limited by the current capacity of the Hartmann layers. Since the current capacity is limited, the Lorentz force is likewise small and the velocity profile is not greatly influenced.

Calculations for the case of insulated top and side walls were carried out and a plot showing the axial velocity distribution as a three dimensional surface is presented for M = 100 in Figure 3. The number of cells in both the y and z directions is increased to 50 and all walls have non-slip velocity conditions.

In practical devices, such as nuclear reactor cooling blankets, duct walls often have non-zero conductivity which allows a return current path outside the fluid. In this section the case of conducting top and bottom walls is studied.

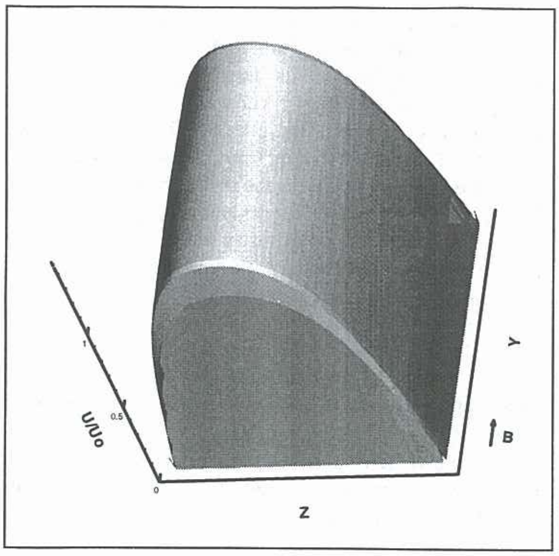


**Figure 3 :** Axial velocity profile with no wall conduction

Calculated axial velocity distribution in the form of a three dimensional surface and current density vector are shown in Figures 4 and 5. Current flow in the z-direction through the core can be significant as the conducting top and bottom walls provide the main return current path in the negative z-direction. As visible in Figure 5, the result is significant y-direction current flows near the insulating side walls. These currents are parallel to the magnetic field and thus make no contribution to the Lorentz force. However currents in the core are perpendicular to the magnetic field giving rise to a Lorentz force which opposes fluid motion in the core region. With a retarding force acting on the fluid in the core the velocity tends to be reduced. Near the insulated side walls a much smaller Lorentz force is acting on the fluid, allowing fluid near the walls to accelerate. This balance of forces produces the characteristic M-shaped velocity profile with the high velocity regions evident in Figure 4. Work is currently under way to validate these predictions against physical results.

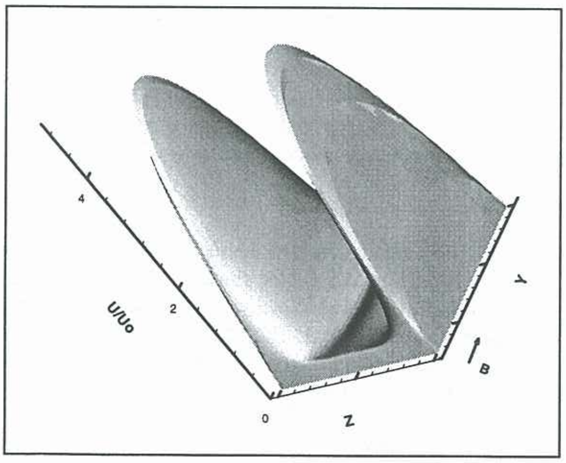


**Figure 4 :** Axial velocity profile for M=100 with conducting top and bottom-walls.

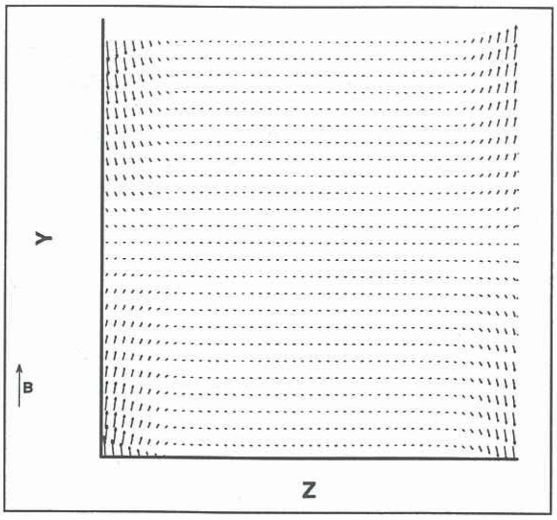


**Figure 5 :** Current density vectors for M=100 and conducting top and bottom walls.

Having demonstrated the model's ability to solve laminar single phase MHD flows, the model can be extended to include turbulent effects via inclusion of the k-ε or Reynolds stress model. The multiphase model in CFX also allows extension of the model to multiphase and free surface problems such as the Hall-Heroult aluminium cell.

## CONCLUSION

The additional equations required to solve Magneto-hydrodynamic problems have been added into the computational fluid dynamics program, CFX. The vector and scalar potential method of solving Maxwell's equations is adopted to obtain the electric and magnetic fields and the resulting Lorentz force.

The model was tested by predicting velocity profiles in a two dimensional duct under the influence of transverse magnetic and electric fields. Predictions were found to accurately match the analytic solution for the problem. The more complex three dimensional case with both insulating and conducting duct walls was examined. For this case the model was able to predict the Hartmann layer and M-shaped velocity profile with high velocity regions near the walls.

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