SOLUTION OF CONVECTION - DIFFUSION EQUATION ON A TRIANGULAR MESH

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ABSTRACT

This paper presents a control volume based finite volume method to solve the convection - diffusion equation on a triangular mesh. Variables are stored at the cell center. The upwind differencing scheme is employed to approximate the convective term. The diffusive flux is evaluated by expressing the normal gradient to the face, in terms of surrounding cell values. The scheme is applied to some test cases and the results are compared with either the exact solutions or the solutions obtained using the quadrilateral meshes, and is shown to perform satisfactory.

INTRODUCTION

Numerical methods are used to study heat transfer, fluid mechanics and other engineering problems. The general concept behind numerical method is to provide an approximate solution to the partial differential equations that describe the physical processes of interest.

Numerical algorithms for the solution of governing fluid flow and heat transfer equations on unstructured meshes have been one of the main focus of CFD research in recent years since unstructured meshes offer more geometric flexibility. Most of the algorithms developed in the recent years are based on finite element method but a few are based on control volume method.

Here we are presenting a control volume based finite volume scheme for unstructured triangular meshes, to solve the two-dimensional convection diffusion equation. The scheme presented by D. J. Mavriplis (7) doesn't couple the neighbor cells with the primary cell, but the scheme presented here couples the neighbor cells well with the primary cell.

GOVERNING EQUATION

The equation for conservation of energy is,

$$\rho c_p \frac{\partial \phi}{\partial t} + \rho c_p u \frac{\partial \phi}{\partial x} + \rho c_p v \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial x}\left(k\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial \phi}{\partial y}\right) \quad (1)$$

NUMERICAL METHOD

Finite Volume Equations

Integration and discretization of the energy equation about a triangular control volume yields

$$A\frac{\phi_p^{n+1} - \phi_p^n}{\Delta t} + \sum_f G_f \phi_f - \alpha \sum_f D_f = 0 \quad (2)$$

Where A is the area of the triangular cell, G_f is the mass flow rate, D_f is the transport due to diffusion through the face f and the summations are over the faces of the control volume.

Modeling the Diffusion Term

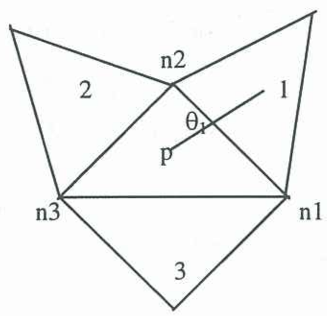

Figure 1: Triangular control volume for energy equation.

The diffusion term at the face is

$$D_f = \left(\frac{\partial \phi}{\partial n}\right)_f l_f \quad (3)$$

Where $(\partial \phi/\partial n)_f$ is the normal gradient to face f and l_f is the length of face f.

Therefore for the triangular control volume shown in Figure 1, the diffusion term could be written as

$$\sum_{f=1}^{3} D_f = \left(\frac{\partial \phi}{\partial n}\right)_{12} l_{12} + \left(\frac{\partial \phi}{\partial n}\right)_{23} l_{23} + \left(\frac{\partial \phi}{\partial n}\right)_{31} l_{31} \quad (4)$$

Consider face n_1n_2.

$$\left(\frac{\partial\phi}{\partial l}\right)_{1p} = \left(\frac{\partial\phi}{\partial n}\right)_{12}\sin\theta_1 + \left(\frac{\partial\phi}{\partial l}\right)_{12}\cos\theta_1 \qquad (5)$$

Where

$(\partial\phi/\partial a)_{p1}$ is the gradient along p1

$(\partial\phi/\partial l)_{12}$ is the gradient along n_1n_2

θ is the angle between 1p and n_1n_2

By rearranging the terms, the normal gradient to face n_1n_2 could be written as

$$\left(\frac{\partial\phi}{\partial n}\right)_{12} = \left(\frac{\partial\phi}{\partial l}\right)_{1p}\frac{1}{\sin\theta_1} - \left(\frac{\partial\phi}{\partial l}\right)_{12}\frac{1}{\tan\theta_1} \qquad (6)$$

By writing the normal gradients to other two faces in the similar manner and discretizing the terms would yield

$$\sum_{f=1}^{3} D_f = \left(\frac{\phi_p-\phi_1}{l_{1p}\sin\theta_1} - \frac{\phi_{n2}-\phi_{n1}}{l_{12}\tan\theta_1}\right) + \left(\frac{\phi_p-\phi_2}{l_{2p}\sin\theta_2} - \frac{\phi_{n3}-\phi_{n2}}{l_{23}\tan\theta_2}\right) + \left(\frac{\phi_p-\phi_3}{l_{3p}\sin\theta_3} - \frac{\phi_{n1}-\phi_{n3}}{l_{31}\tan\theta_3}\right) \qquad (7)$$

ϕ values at n_1, n_2 and n_3 are assumed as the weighted average of ϕ values at the centers of the neighbor cells.

Modeling the convection term

The convection term at the face is

$$G_f\phi_f = (u\,\bar{i} + v\,\bar{j}).\bar{n}\, l_f \qquad (8)$$

Where G_f is the mass flow rate through the face

$\bar{n}$ is the outward normal vector.

Therefore for the triangular control volume shown in Figure 1, the convective term could be written as

$$\sum_{f=1}^{3} G_f\phi_f = G_{12}\phi_{12} + G_{23}\phi_{23} + G_{31}\phi_{31} \qquad (9)$$

Since the velocity field is known, the task of evaluation of the convective flux reduces to determining the face value ϕ_f. By using the first order upwind differencing scheme,

$$\phi_f = \phi_{upwind} \qquad (10)$$

Discretized Equations

The discretization procedure yields the following linear system of equations for ϕ at the cell centers.

$$a_p\phi_p = a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + Su \qquad (11)$$

Where

ϕ_p - value of ϕ at the cell center.

ϕ_1, ϕ_2 and ϕ_3 - values of ϕ at centers of neighbor cells.

Su - Source term

RESULTS

In this section, the formulation developed above is applied to the problem of the flow of a cold fluid entering a hot two-dimensional domain, with two different flow conditions. Away from the sidewalls of the domain the temperature distribution is two dimensional, and governed by equation (1). For the first case, the solution is compared with the solution obtained using quadrilateral mesh and for the second case, the solution is compared with the analytical solution.

1. Flow in a two dimensional duct with parabolic velocity profile

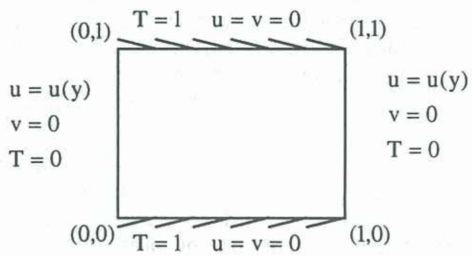

Figure 2: Computational domain

For the computational domain shown in Figure 2, the boundary conditions are

$$\begin{aligned} T(0,y) &= 0 \\ \partial T/\partial x &= 0 \quad \text{at } x = 1 \\ T(x,0) &= 1 \\ T(x,1) &= 1 \end{aligned} \qquad (12)$$

The velocity profile is known and is given by,

$$u(y) = 2*[0.5^2-(y-0.5)^2] \qquad (13)$$

$$v = 0$$

The steady state solution along the centerline is compared with the solution obtained using the quadrilateral mesh.

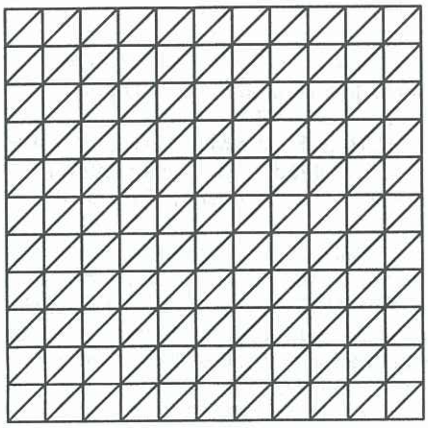

Figure 3: Triangular Mesh arrangement for both cases

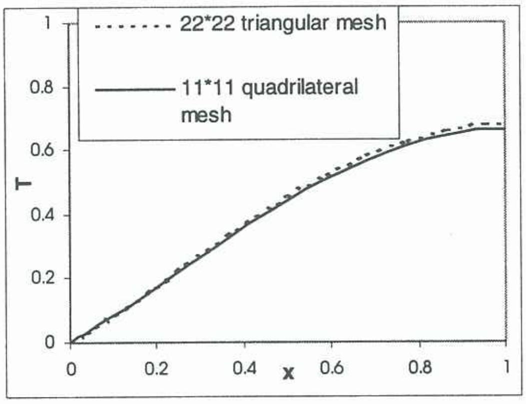

Figure 4: Temperature along the centerline

The results presented here shows that the solution obtained using the triangular mesh agrees well with the solution obtained using the quadrilateral mesh.

2. Flow in a two dimensional domain with exponentially varying boundary temperatures

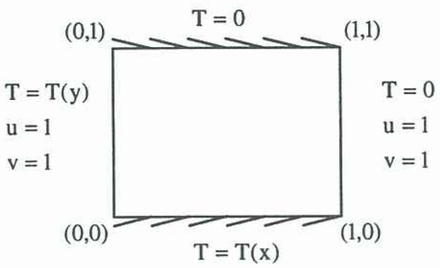

Figure 5: Computational domain

For the computational domain shown in Figure 5, the boundary conditions are

$$T(x,0)=\frac{1-\exp[(x-1)u/\alpha]}{1-\exp(-u/\alpha)} \qquad T(x,1)=0 \qquad (14)$$

$$T(0,y)=\frac{1-\exp[(y-1)v/\alpha]}{1-\exp(-v/\alpha)} \qquad T(1,y)=0$$

The velocity profile is known and is given by

$$u = 1.0 \qquad (15)$$
$$v = 1.0$$

The steady state solution along the centerline is compared with the analytical solution.

The analytical solution (derived by Rai, 1982) for the above problem, is given by

$$T(x,y)=\left\{\frac{1-\exp[(x-1)u/\alpha]}{1-\exp(-u/\alpha)}\right\}\left\{\frac{1-\exp[(y-1)v/\alpha]}{1-\exp(-v/\alpha)}\right\} \qquad (16)$$

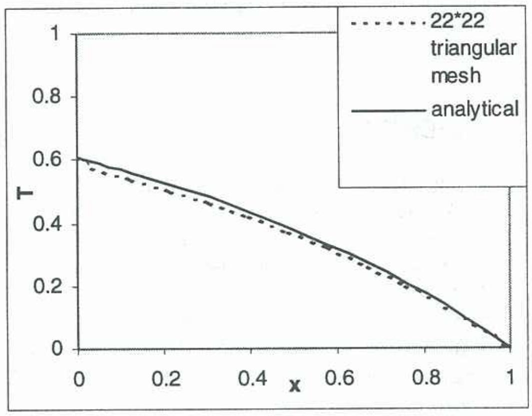

Figure 6: Temperature along the centerline

The results presented here shows that the solution obtained using the triangular mesh agrees well with the analytical solution.

CONCLUSION

A control volume method based finite element method for the solution of two-dimensional convection - diffusion equation, using the triangular mesh, is described. Encouraged by the success of this formulation, efforts are being made to extend this method to solve the coupled fluid flow equations.

NOMENCLATURE

A	Area of the triangular control volume
c_p	Specific heat at constant pressure
D_f	Diffusive flux on face f
G_f	Mass flux on face f
i, j	Unit vectors in the x, y directions
k	Thermal conductivity
l	Length of the face
$\bar{n}$	The unit normal to the cell face
n1, n2 and n3	Node numbers of the triangular cell
Su	Source term
T	Temperature
Δt	Time step
u, v	Cartesian velocity components
x, y	Cartesian coordinates
α	Diffusion coefficient
ρ	Density
ϕ	Transported scalar
ϕ_P	Value of ϕ at the cell center
ϕ_1, ϕ_2 and ϕ_3	Values of ϕ at the cell centers of neighbor cells

REFERENCES

ANDERSON, D. A., TANNEHILL, J. C. and PLETCHER, R. H., "Computational Fluid Mechanics and Heat Transfer", Hemisphere Publishing Corporation, New York, 1984.

BARBARA LEDAIN MUIR and RABI BALIGA, "Solution of Three-Dimensional Convection-Diffusion Problems using Tetrahedral Elements and Flow-Oriented Upwind Interpolation Functions", Numerical Heat Transfer, **9**, 143-162, 1986.

CHOW, P., CROSS, M. and PERICLEOUS, K., "A natural extension of the conventional finite volume method into polygonal unstructured meshes for CFD application", Applied Mathematical Modeling, **20,** 170- 183, 1996.

DARTZI PAN, CHO-HSIN LU and JEN-CHIEH CHENG, "Incompressible Flow Solution on Unstructured Triangular Meshes", Numerical Heat Transfer, Part B, **26**, 207-224, 1994.

DESPOTIS, G. K. and TSANGARIS, S., "Fractional Step Method for Solution of Incompressible Navier-Stokes Equations on Unstructured Triangular Meshes, International Journal for Numerical Methods in Fluids, **20**, 1273-1288, 1995.

FRINK, NEAL T., "Upwind scheme for solving the Euler equations on Unstructured Tetrahedral Meshes", AIAA Journal, **30-1**, 70-77, 1992.

Mavriplis, D. J., "A Three Dimensional Multigrid Reynolds-Averaged Navier-Stokes Solver for Unstructured Meshes", NASA Contractor Report 194908, ICASE Report No. 94-29, 1-26, May 1994.

KAYS, W. M. and CRAWFORD, M. E., "Convective Heat and Mass Transfer", McGraw - Hill, New York, 1993.

MATHUR, S. R. and MURTHY, J. Y., "A Pressure Based Method for Unstructured Meshes, Numerical Heat Transfer, Part B, **31**, 195-215, 1997.

PATANKAR, S. V., "Numerical Heat Transfer and Fluid Flow", McGraw - Hill, New York, 1980.

THOMADAKIS, M. and LESCHZINER, M., "A Pressure - Correction Method for the Solution of Incompressible Viscous Flows on Unstructured Grids", International Journal for Numerical Methods in Fluids, **22**, 581-601, 1996.