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INSTABILITY OF FLOWS WITH VERTICAL AND LONGITUDINAL TEMPERATURE GRADIENTS FOR DIFFERENT BOUNDARY CONDITIONS

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ABSTRACT

The thermal instability of a general class of horizontal flows comprising forced and temperature-induced flow components as well as temporal temperature changes is studied. The instability is usually triggered by an unstable layer of height h_u, which arises as a consequence of the nonlinear vertical temperature profile typical for such flows. The effects that the different components have on the flow's stability are quite different for the two sets of boundary conditions considered here (constant heat flux and prescribed temperature). Some results of a linear stability analysis are also given.

INTRODUCTION

This paper is motivated by recent results of experiments in a side-heated cavity (Schöpf and Stiller, 1997). Briefly, when a rectangular, fluid-filled container is suddenly heated from one of its vertical sidewalls, a convective upward motion starts near that sidewall. The flow is confined to a narrow boundary layer and, upon reaching the top of the cavity, it exits into a horizontal intrusion. Thus, heated fluid intrudes along the cavity ceiling into the still isothermal fluid in front of it. Similarly, the sudden cooling of one of the sidewalls would result in an intrusion flow across the cavity floor (for more details, see Patterson and Imberger, 1980).

Previously it has been assumed that the cavity flow is essentially two-dimensional, with the only variation in the third, cross-stream dimension being a trivial flow adjustment near the front and end plates of the cavity (see e.g. Schladow, 1990; Armfield and Patterson, 1992; Schöpf and Patterson, 1995). The abovementioned experiments, however, have clearly shown that the cavity intrusion flow undergoes a Rayleigh–Bénard-type instability leading to longitudinal convection rolls, i.e. rolls parallel to the flow direction. Direct experimental evidence of those rolls has been obtained by employing the shadowgraph technique.

An example of such a shadowgraph image is shown in Fig. 1, where the horizontal intrusion flow is seen from above. The heated wall is at $x = 0$, so that the intrusion advances in the positive x-direction, i.e. from bottom to top in this representation. The bright, almost parallel lines are caused by a regular temperature variation along the y-direction and correspond to downwelling cold fluid, with layers of upwelling warmer fluid between them. For more details of the experiments, see Schöpf and Stiller (1997).

The aim of this contribution is to give a theoretical explanation for this instability.

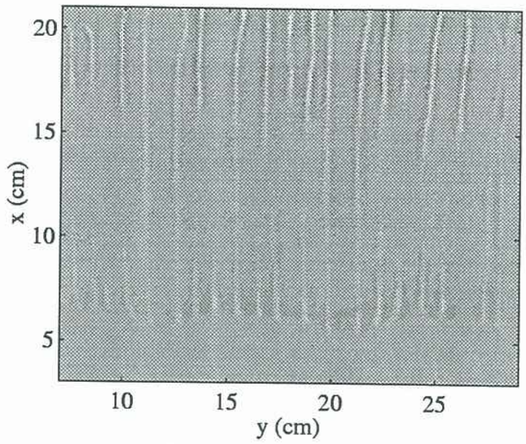

Figure 1: Shadowgraph image of the intrusion flow when illuminated from below.

BOUNDARY CONDITIONS, BASIC EQUATIONS

We use a simplified model of a parallel flow between horizontal plates. The flow is assumed to be of infinite horizontal extent in the x–y-plane and is subjected to a horizontal pressure gradient Δp_x. In the vertical z-direction, it is bounded by two no-slip boundaries (at $z = \pm H$). We consider two sets of thermal boundary conditions, namely constant heat flux boundaries and boundaries with prescribed temperatures.

The Boundary Conditions

In the first case, the heat fluxes $\partial_z T(z = H) = \Theta_u$ at the upper and $\partial_z T(z = -H) = \Theta_l$ at the lower boundary are held constant. The heat exchange with these boundaries leads to a constant horizontal temperature gradient β.[1] When the horizontal heat advection is not balanced by the vertical heat flux through the boundaries, the flow exhibits a net horizontal heat transport leading to internal heating, which causes the isotherms to move with a constant velocity U_0 (see below; for more details, see Stiller and Schöpf, 1997).

In the second case, the temperatures at the boundaries are prescribed in the form $T(z = \pm H) = \mp\Delta T/2+\beta x+c_b t+const$. These boundaries impose a vertical temperature difference ΔT, a horizontal temperature gradient β, and temperature changes in time with a constant rate c_b. Defining $U_0 = -c_b/\beta$ and using an appropriate non-dimensionalization, the boundary condition can be written as $T(z = \pm 1) = \mp\Delta T/2+\beta(x-U_0 t)+const.$, showing that the time dependence of the boundaries vanishes when observed from a frame of reference that moves with U_0.

The Basic Equations

Time and temperature are scaled in units of H, H^2/ν and $\nu^2/(g\alpha H^3)$, respectively. H is half the channel height, g is the acceleration due to gravity, α is the coefficient of thermal expansion and ν is the kinematic viscosity. In the following, we will consider parallel flows of the general form

$$U = \bar{U}(z), \quad V = W = 0\,, \tag{1}$$
$$T = \bar{T}(z) + \beta(x - U_0 t)\,, \tag{2}$$

where U and V are the horizontal and W the vertical velocity components. The profiles $\bar{U}(z)$ and $\bar{T}(z)$ have to be derived from the Navier–Stokes equations and the heat equation, which in the Boussinesq approximation read

$$(\partial_t + U\partial_x + W\partial_z)\,U = -\partial_x P + \nabla^2 U\,, \tag{3}$$
$$(\partial_t + U\partial_x + W\partial_z)\,W = -\partial_z P + \nabla^2 W + T\,, \tag{4}$$
$$Pr\,(\partial_t + U\partial_x + W\partial_z)\,T = \nabla^2 T\,. \tag{5}$$

$Pr = \nu/\kappa$ is the Prandtl number with κ being the thermal diffusion.

Inserting Eqs. (1,2) into Eq. (4) yields $0 = -\partial_z P + T$ and thus $\partial_x P = \int \partial_x T dz + \Delta p_x = \beta z + \Delta p_x$. By using this to eliminate the pressure from Eq. (3) and then inserting Eqs. (1,2) into Eqs. (3,5), we finally obtain coupled equations for $\bar{U}(z)$ and $\bar{T}(z)$

$$\partial_z^2 \bar{U} = \beta z + \Delta p_x\,, \tag{6}$$
$$\partial_z^2 \bar{T} = Pr\beta(\bar{U} - U_0)\,. \tag{7}$$

THE BASE FLOW

The velocity field can be derived from Eq. (6) as a superposition of a temperature-induced flow proportional to β (and with zero average) and a Poiseuille flow forced by the constant horizontal pressure gradient Δp_x (with an average velocity $v_m = -\Delta p_x/3$):

$$\bar{U}(z) = \beta U_\beta(z) + v_m U_p(z)\,. \tag{8}$$

With the no-slip boundary condition $\bar{U}(z = \pm 1) = 0$ we find $U_\beta(z) = z(z^2-1)/6$ and $U_p(z) = (3/2)(1-z^2)$. Equation (7) for the temperature field has to be solved separately for the different boundary conditions. For constant heat flux boundaries, we find

$$\bar{T}(z) = Pr\beta v_m\left(T_p(z) + \frac{\beta}{v_m}T_\beta^{cf}(z) - \frac{U_0}{v_m}\frac{z^2}{2}\right) + \frac{\Theta_l + \Theta_u}{2}z\,, \tag{9}$$

while for boundaries with prescribed temperatures

$$\bar{T}(z) = Pr\beta v_m\left(T_p(z) + \frac{\beta}{v_m}T_\beta^{pt}(z) - \frac{U_0}{v_m}\frac{z^2}{2}\right) - \frac{\Delta T}{2}z \tag{10}$$

with

$$\partial_z^2 T_p = U_p(z)\,, \quad \partial_z^2 T_\beta^{cf} = \partial_z^2 T_\beta^{pt} = U_\beta(z)\,. \tag{11}$$

Choosing $T_p(+1) = T_p(-1)$ determines the temperature profiles in Eqs. (9,10) up to a physically irrelevant constant, and we find $T_p(z) = z^2(3 - z^2/2)/4$, $T_\beta^{cf}(z) = z(z^4/10 - z^2/3 + 1/2)/12$, and $T_\beta^{pt}(z) = z(z^4/10 - z^2/3 + 7/30)/12$. Therefore, T_β^{cf} and T_β^{pt} for the different boundary conditions differ only by a term $z/45$.

SOME RESULTS

Depending on the boundary condition and the various parameters β, v_m, U_0 in Eqs. (9,10), a variety of different solutions with different characteristics is possible. In the following, we will discuss the thermal stability properties for some of the most interesting cases (for more details, see Stiller and Schöpf, 1997 and Stiller et al., 1998). For simplicity, β and v_m are assumed to be positive, which typically leads to a potentially unstable layer of height h_u near the bottom boundary.[2]

The Influence of the Temperature-Induced Flow

In order to discuss the influence of the term $T_\beta(z)$ in Eqs. (9,10), we assume a stationary flow with $U_0 = 0$. Various solutions for this case are shown in Fig. 2, where (a) refers to the constant heat flux boundary condition and (b) to the boundaries with prescribed temperatures. The respective left parts of

[1] One may think of a cold fluid which is heated by the boundaries as it flows through an imperfectly insulated channel. In particular, this covers situations where both boundaries are heated.

[2] A imultaneous sign change of both β and v_m simply turns a right-flowing flow into a left-flowing flow, while a sign change of either β or v_m shifts the unstable layer from the bottom to the top boundary.

Figs. 2(a,b) show various velocity fields and are identical, since $\bar{U}(z)$ is independent of the thermal boundary condition (see Eq. 8). The two middle parts show the temperature components $T_\beta^{cf}(z)$ and $T_\beta^{pt}(z)$ induced by U_β (their analytical forms are given after Eq. 11). The respective right parts of Figs. 2(a,b) show the resulting temperature fields for the two different sets of boundary conditions. The different curves in the left and right parts correspond to the different values of $\beta/v_m \ll 1$ for the bold solid lines, and $\beta/v_m = 8, 16, 32$ for the dotted, dashed and long-dashed lines, respectively. For sufficiently large β/v_m, we find $\bar{U}(z) < 0$ in the top region of the channel, indicating a backflow (see left parts of Fig. 2).

For the case of constant heat flux boundaries, it turns out that the temperature component T_β^{cf} has an entirely stabilizing effect on $\bar{T}$ (see middle part of Fig. 2a). Note, that an increase of β/v_m increases the contribution of T_β^{cf} to the temperature profile (see Eq. 9). Therefore, neglecting β/v_m in the analysis could mean to underestimate the thermal stability of the flow, and flows with a given net vertical heat flux are stable for sufficiently small v_m. The right part of Fig. 2(a) shows how the resulting temperature profiles become less unstable for increasing values of β/v_m, i.e. for a large influence of $T_\beta^{cf}(z)$.

The case of boundaries with prescribed temperatures is illustrated in Fig. 2(b). Here, the derivative of T_β^{pt} is positive only in the middle part of the channel and negative near the boundaries (see middle inset of Fig. 2b). Correspondingly, an increase of β/v_m is destabilizing when h_u is small or zero (the case shown in the right subset of Fig. 2b), whereas it has a stabilizing effect when the potentially unstable layer extends well over the middle part of the channel. In the latter case, it decreases the height h_u of the unstable layer as well as the temperature difference ΔT_u across it (not shown here). This example demonstrates how the β-induced flow can have a very different effect on the stability characteristics for different boundary conditions.

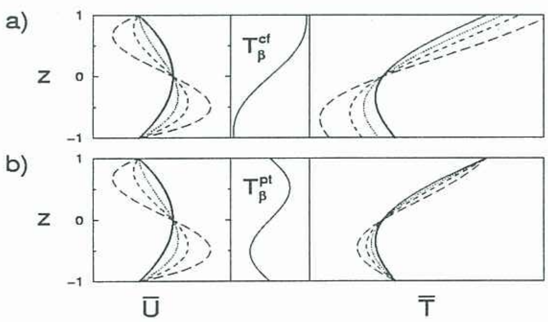

Figure 2: Velocity profiles (left) and temperature profiles (right) for different values of β/v_m, with a potentially unstable layer being obvious near the bottom boundary. (a) Constant heat flux boundaries. (b) Boundaries with prescribed temperatures.

The Influence of Temporal Temperature Variations

The parameter U_0 can be identified as the velocity of the isotherms of the temperature field. In order to study its influence, we assume $\beta/v_m \ll 1$, so that the β-induced terms with T_β^{cf} and T_β^{pt} in Eqs. (9,10) can be neglected and the flow is essentially of the Poiseuille type. U_0 has no effect on the velocity profile given by Eq. (8), however, it alters the vertical temperature profile by adding a term proportional to $-z^2/2$ (see Eqs. 9,10).

The case of constant heat flux boundaries is illustrated in Fig. 3(a), where the flow is above an insulated wall ($\Theta_l = 0$) and heated from the top boundary ($\Theta_u > 0$). The different temperature profiles on the right correspond to different values of U_0, with $U_0/v_m = 0, 0.4, 0.7, 0.8, 0.9$ from right to left. For the stationary case $U_0 = 0$, the boundary condition forces $\bar{T}'(z) > 0$ everywhere. For $U_0 > 0$, however, there is a potentially unstable region with $\bar{T}'(z) < 0$ at the bottom which becomes more unstable for increasing U_0, even though the flow as a whole is heated from above. Near the bottom wall, the heat advection is negative when observed from the frame of reference moving with the isotherms, so that the unstable layer is the result of hot fluid being advected into the bottom region.

A typical situation for the prescribed boundary temperature case is shown in Fig. 3(b), where again the fluid as a whole is heated from above. The different temperature profiles correspond from left to right to $U_0/v_m = -0.8, -0.4, -0.2, 0, 0.2, 0.4, 0.8$. The term proportional to $-z^2/2$ has a stabilizing effect in the lower and a destabilizing effect in the upper part of the channel. Since the flows considered here have their unstable region at the bottom of the channel, it is the behaviour in this lower region which determines the instability, at least for not too large h_u. Therefore, $U_0 < 0$ is generally destabilizing and $U_0 > 0$ is stabilizing (see Fig. 3b), except for very large values of U_0/v_m when the term proportional to $-z^2/2$ creates an unstable region in the upper part of the channel (not shown here). In Fig. 3(b), the unstable layer for $U_0 = 0$ (bold solid line) disappears for sufficiently large $U_0/v_m > 0$.

This behaviour is quite different from the effect for constant heat flux boundaries. In order to gain an intuitive understanding, note that for constant heat flux boundaries, situations with $\beta U_0 > 0$ may result from linearly decreasing the fluid temperature at the inflow of the channel so that the time dependence of the temperature inside the channel is a consequence of heat advection (bulk heating) alone. Therefore, the bottom layer is in some way effectively cooled from *above*, since the advective heat transport is stronger in the middle of the channel where the flow velocity is larger. For conducting boundaries, on the other hand, $\beta U_0 > 0$

is caused by decreasing the temperature at the boundaries which means an effective cooling from *below*, since the bottom boundary will have a larger impact on the bottom layer than the upper boundary.

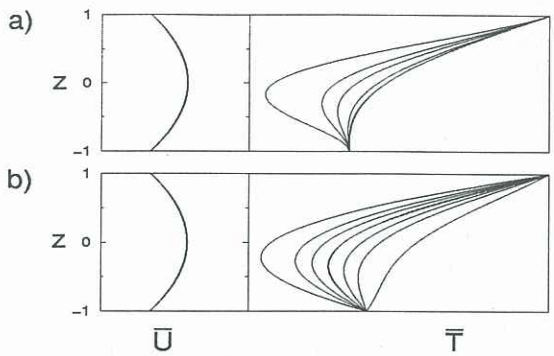

Figure 3: Vertical temperature profiles (right) for different values of U_0/v_m for a Poiseuille flow (left). (a) Constant heat flux boundaries. (b) Boundaries with prescribed temperatures.

LINEAR STABILITY ANALYSIS

For the base flows discussed above, a linear stability analysis has been performed, the details of which can be found in Stiller and Schöpf (1997) and Stiller et al. (1998). The most interesting results are as follows.

It is useful to define the Rayleigh number with respect to the height h_u of the unstable layer and the temperature difference ΔT^u across it, i.e. $Ra^u = g\alpha\Delta T^u h_u^3/\kappa\nu$. Similarly, the wavenumber is given by $k^u = kh_u$. It turns out that as h_u decreases, the critical values Ra_c^u and k_c^u become independent of h_u. Generally, the critical values Ra_c^u for the onset of convection are much smaller than the value of 1707 that is associated with the classical Rayleigh–Bénard system. For $h_u \ll 2$, i.e. a potentially unstable layer that is much thinner than the channel height, we find $Ra_c^u \approx 200\ldots300$ for the constant heat flux boundaries and $Ra_c^u \approx 500$ for the prescribed temperature boundaries. For the critical wavenumber, we find $k_c^u \approx 1\ldots1.5$ and $k_c^u \approx 2$, respectively, as compared to 3.116 for the Rayleigh–Bénard case. These values model the experimental results very well, where the overall temperature difference would not be sufficient for a normal Rayleigh–Bénard instability (see Schöpf and Stiller, 1997).

CONCLUSION AND OUTLOOK

We have investigated the thermal instability of horizontal, parallel flows between conducting boundaries and between boundaries with constant heat fluxes. These flows include many features that are inherent to more complicated systems, however, due to their simplicity they allow a systematic study and understanding. Most prominent is the effect of horizontal heat advection which may lead to an unstable layer with $\partial_z\bar{T} < 0$ and thus trigger an instability.

Although formally similar, the flow types for the different boundary conditions were found to behave quite differently in many respects. We have shown that for boundaries with prescribed temperatures, the β-induced flow component may be either stabilizing or destabilizing, while it is always stabilizing for constant heat flux boundaries. Temporal variations in the boundary temperature corresponding to $U_0 \neq 0$ were found to have the opposite effect for the two cases. These results underline the different nature of these flows which arise in quite a different context.

For a more complete picture, the effect of the nonlinear temperature profiles on shear-driven instabilities has to be studied as well. In some closed flows ($v_m = 0$), such transverse modes were in fact found to be the most unstable ones. We finally note that the stability results discussed here were obtained for flows with a Prandtl number of order 7, as apropriate for water. Smaller Prandtl numbers are expected to have a significant effect on the stability properties and will be discussed elsewhere.

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