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DUAL STREAM FUNCTIONS FOR 3D SWIRLING FLOWS

Andrei V. REZTSOV and Gordon D. MALLINSON

Department of Mechanical Engineering
The University of Auckland, Auckland, NEW ZEALAND

ABSTRACT

Sets of dual stream functions that can be used to describe three dimensional vector fields, typical of those in swirling flows, are described. The particular interest is those situations where one of the stream functions is multi-valued. The ways that they can be used to represent well known idealised flows is presented with the purpose of demonstrating their potential for application to the visualisation of three dimensional flows.

INTRODUCTION

Dual stream functions, f and g, defined by

$$\mathbf{m} = \nabla f \times \nabla g \tag{1}$$

can be used to describe a three dimensional solenoidal vector field, **m**. Although these functions and their relationships with stream surfaces are well known, their use in fluid mechanics is relatively limited.

Recently, their use in visualisation has been explored. Intersections of pairs of iso-surfaces define stream lines. Kenwright and Mallinson (1992), and Knight and Mallinson (1996), used the mapping from physical to stream function space to simplify and speed up stream line construction. Beale (1997) obtained global stream functions that were used to construct stream contours.

Others such as Greywall (1993), Normandin and Clearemont (1996) and Keller (1996), have used them to solve the equations of fluid motion. The stream functions define coordinate surfaces that contain the local velocity vector. The surface normals along with the velocity vector define a three-dimensional coordinate system in which the transformed equations of motion can be recast and used to find a coordinate 'mesh' which is aligned with the flow.

Application of the dual stream function approach has been limited to situations where both stream functions are single valued. Stream line construction is restricted to 'local' methods for advancing through computational cells. Computational methods for constructing flow aligned meshes have been limited to non-recirculating flows, such as extrusions, parabolic flows in ducts and potential flow in diffusers.

The work described here has the long term objective of developing a theoretical basis for using dual stream functions to visualise and model recirculating flows and ultimately general three-dimensional flows. Whereas the conventional approach has been to assign equal importance to the two functions, this work associates one with a 'structural' significance. In mathematical terms the structural function, if it exists, is single valued. The other function is, in general, multi-valued.

MOTIVATION

The nature of the problem can be illustrated by fluid motion that occurs in a cavity which has one vertical surface heated and the opposite surface cooled. Stream lines in this cavity follow spiralling paths which appear to lie in closed surfaces (Figure 1). Although these surfaces are apparent in the diagram, there is no known method for generating them *a priori*.

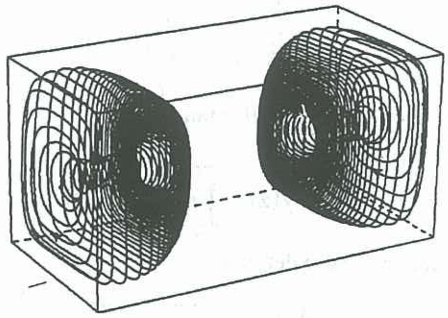

Figure 1: Stream lines in a closed cavity heated on one of the large vertical surfaces and cooled on the other.

This problem is being approached from two directions. The first seeks to develop methods for identifying the single valued surface from numerical descriptions of vector fields. The second explores the mathematical basis of the relationships between the functions and the fields they describe by deriving analytical functions for well known situations. In this paper the second direction has been taken. Mathematical forms that extend the known analytical dual stream functions are described. Their use and relationship with flow fields are demonstrated for flows near singularities identified by Reyn (1964) and Chong Perry and Cantwell (1990).

DUAL STREAM FUNCTIONS FOR SWIRLING ELLIPTICAL FLOWS

The starting point for the derivation of the functions described herein was to consider a trajectory

$$\mathbf{p} = \big(\psi(\varphi(t))a\cos\omega t,\quad \psi(\varphi(t))b\sin\omega t,\quad \varphi(t)\big) \tag{2}$$

that describes elliptical paths with varying dependence of z on t. The function ψ determines the dependence of the 'size' of the path on z. The velocity on the trajectory (2) is

$$\begin{aligned} \mathbf{v} &= (u, \quad v, \quad w) \quad \text{where} \\ u &= \dot{\psi} a \cos\omega t - \psi a \omega \sin\omega t \\ v &= \dot{\psi} b \sin\omega t + \psi b \omega \cos\omega t \\ w &= \dot{\varphi} \end{aligned} \tag{3}$$

(where $\dot{\psi} = \dfrac{d\psi}{d\varphi}\dot{\varphi} = \psi'\dot{\varphi}$)

In general, **v**, may be non-solenoidal. In order for it to be recovered from a set of dual stream functions, it can be related to a solenoidal field, **m**, by

$$\mathbf{m} = \rho \mathbf{v} \tag{4}$$

where **m** is described by dual stream functions via (1). In this paper, the 'density', ρ, will be assumed to be a function of z.

Stream functions for (4) are

$$g = \frac{abr^2}{2\psi(z)^2} \tag{5}$$

$$f = \omega\gamma(z) - c\theta \tag{6}$$

where

$$r = \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2} \tag{7}$$

$$\theta = \tan^{-1}\left(\frac{ay}{bx}\right) \tag{8}$$

$$\gamma(z) = \int_{z_0}^{z} \rho(\tau)\psi^2(\tau)d\tau \tag{9}$$

From the above definitions

$$\nabla g = \frac{1}{\psi^2}\left(\frac{xb}{a}, \quad \frac{ya}{b}, \quad -\frac{ab\psi' r^2}{\psi}\right) \tag{10}$$

$$\nabla f = \left(\frac{cy}{abr^2}, \quad \frac{-cx}{abr^2}, \quad \omega\rho\psi^2\right) \tag{11}$$

These lead to

$$\begin{aligned} \mathbf{m} &= \nabla f \times \nabla g \\ &= \left(\frac{cx\psi'}{\psi^3} - \frac{\omega\rho ya}{b}, \quad \frac{cy\psi'}{\psi^3} + \frac{\omega\rho xb}{a}, \quad \frac{c}{\psi^2}\right) \end{aligned} \tag{12}$$

As can be readily verified, $\nabla . \mathbf{m} = 0$.

The velocity is

$$\begin{aligned} \mathbf{v} &= \frac{\mathbf{m}}{\rho} \\ &= \left(\frac{cx\psi'}{\rho\psi^3} - \frac{\omega ya}{b}, \quad \frac{cy\psi'}{\rho\psi^3} + \frac{\omega xb}{a}, \quad \frac{c}{\rho\psi^2}\right) \end{aligned} \tag{13}$$

The divergence of the velocity field is

$$\nabla . \mathbf{v} = -\frac{c\rho'}{\rho^2\psi^2} \tag{14}$$

In particular, if the density is chosen to be

$$\rho = \frac{c}{w\psi^2} \tag{15}$$

γ is independent of ψ and the velocity field is described by

$$\mathbf{v} = \left(\frac{xw\psi'}{\psi} - \frac{\omega ya}{b}, \quad \frac{yw\psi'}{\psi} + \frac{\omega xb}{a}, \quad w\right) \tag{16}$$

which for $r = \psi$, $w = \dot{\varphi}$ and $\theta = \omega t$ corresponds to the trajectory given by (2).

Special cases of the flows represented by these stream functions will now be considered.

HELICAL FLOW

Perhaps the simplest example is the elliptical helical flow where

$$\rho = 1, \quad \psi = 1 \Rightarrow \gamma = z \tag{17}$$

with stream functions,

$$g = \frac{abr^2}{2} \quad f = \omega z - c\theta \tag{18}$$

The velocity and vorticity are

$$\mathbf{v} = \left(-\frac{\omega ya}{b}, \quad \frac{wxb}{a}, \quad c\right) \tag{19}$$

$$\varsigma = \left(0, \quad 0, \quad \omega\frac{b^2 + a^2}{ab}\right) \tag{20}$$

As Figure 2 illustrates, iso-surfaces of g are elliptical cylinders and those of f are helices. (For clarity, the stream line in the figure is offset from the one formed by the intersection of the two iso-surfaces.)

The stream surfaces are constructed by setting the left hand sides of (18) constant, i.e.,

$$r = \sqrt{\frac{2g_c}{ab}} \quad z = \frac{f_c + c\theta}{\omega} \tag{21}$$

where $0 \le \theta \le 2n\pi$ to construct n cycles of the f surface.

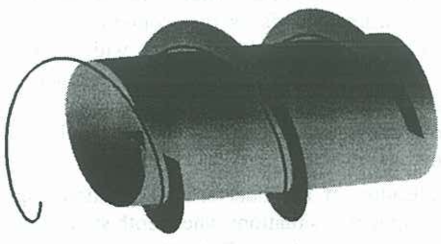

Figure 2: Dual stream functions for elliptical helical flow for a=2, b=1, ω=2, c=5.

EXAMPLES OF FLOWS NEAR SINGULARITIES

Solutions to (5-6) can include flow in the regions of the singularities identified by Chong, Perry and Cantwell (1990). The interest here is that representations near singularities may indicate how stream functions for general three-dimensional recirculating flows might be constructed.

Centre - stretching

Stream lines in the plane of symmetry (z=0) are closed. The z velocity component, w, decreases as the fluid approaches this plane. The velocity field is non-solenoidal but may be constructed with dual stream functions in the following manner.

The structural surfaces are cylinders, hence ψ=1. A suitable choice for w is,

$$w = cz \tag{22}$$

with negative values of c producing flow towards the z=0 plane. This leads to

$$\rho = \frac{c}{w\psi^2} = \frac{1}{z} \tag{23}$$

so that

$$\gamma = \ln(z) \tag{24}$$

The g stream function is the same as for the previous example but the f function is,

$$f = \omega \ln(z) - c\theta \tag{25}$$

The mass flow, and velocity fields are,

$$\mathbf{m} = \left(-\frac{\omega y a}{bz},\ \frac{\omega x b}{az},\ c \right) \tag{26}$$

$$\mathbf{v} = \left(-\frac{\omega y a}{b},\ \frac{\omega x b}{a},\ cz \right) \tag{27}$$

The divergence of **v** is c. Note that the mass flux vector is everywhere parallel to the velocity vector. Hence, although the stream functions are strictly defined for **m** they are also stream surfaces of **v**. The vorticity is the same as for the helical flow.

Example stream functions and stream lines, constructed by numerically integrating (27), are shown in Figure 3.

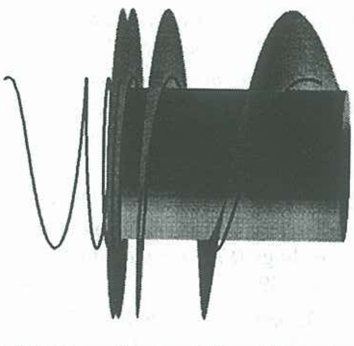

Figure 3. Stream surfaces and lines for a centre-stretching flow for a=2, b=1, ω=5 and c=-1. Two stream lines have been drawn, one from each side of the symmetry plane.

Unstable focus-stretching

This next example involves a non constant ψ.

Stream lines in the plane of symmetry spiral outwards. The structural surfaces are parabolas of revolution so that suitable choices for ψ and w are,

$$\psi = \sqrt{z} \qquad w = cz \quad \Rightarrow \rho = \frac{1}{z^2} \tag{28}$$

leading to stream functions

$$g = \frac{abr^2}{2z} \quad f = \omega \ln(z) - c\theta \tag{29}$$

The resulting velocity field is

$$\mathbf{v} = \left(\frac{cx}{2} - \frac{\omega y a}{b},\ \frac{cy}{2} + \frac{\omega x b}{a},\ cz \right) \tag{30}$$

and the vorticity is the same as for the previous examples. Example stream surfaces and lines are shown in Figure 4.

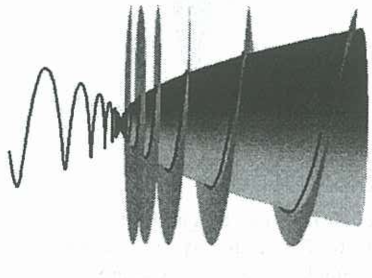

Figure 4. Stream surfaces and lines for an unstable focus flow for a=1, b=1, ω=40 and c=-1. Two stream lines have been drawn, one from each side of the symmetry plane.

Saddle-spiral

This is similar to the one above, but the stream lines asymptotically approach the plane of symmetry (Reyn 1964). A suitable choice for ψ is $\psi = \frac{1}{z}$ and for the same z velocity as used for the previous examples

$$\rho = z \tag{31}$$

The g stream function is

$$g = \frac{z^2 abr^2}{2} \tag{32}$$

It is instructive, however, to consider the solution for ρ=1. The function f is now

$$f = -\frac{\omega}{z} - c\theta \tag{33}$$

The velocity and vorticity are

$$\mathbf{v} = \left(-cxz - \frac{\omega y a}{b},\ -cyz + \frac{\omega x b}{a},\ cz^2 \right) \tag{34}$$

$$\varsigma = \left(cy,\ -cx,\ \omega \frac{b^2 + a^2}{ab} \right) \tag{35}$$

This solution reflects the fact that the flow structure admits naturally to a mass conserving flow approaching the plane. (This occurs at each end of the toroidal flow in Figure 1.) As the example (Figure 5) shows, the stream lines have the correct behaviour. Since w depends on the square of z, the coefficient c had to be reduced to produce an acceptable rendering of the surfaces. The stream function (25) produces a gentler approach into the spiral, albeit, at the expense of producing a velocity field having non zero divergence.

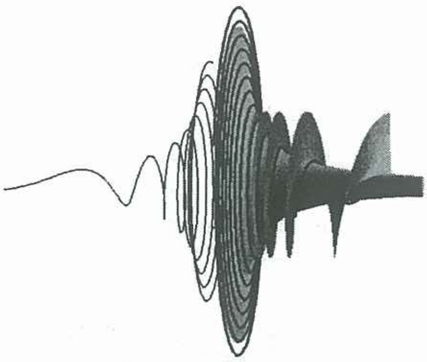

Figure 5. Stream surfaces and lines for a saddle - spiral flow for a=1, b=1, ω=5 and c=-0.25.

TOROIDAL FLOWS

A toroidal flow can be generated using the general functions (5-6). For simplicity the case a=b=1, will be considered. Suitable stream functions are

$$g = \frac{(r-r_0)^2}{2} + \frac{(z-z_0)^2}{2} \qquad f = \omega z - c\theta \tag{36}$$

The velocity field, which has zero divergence, is

$$u = -\frac{c(z-z_0)x}{r^2} - \frac{\omega(r-r_0)y}{r}$$
$$v = -\frac{c(z-z_0)y}{r^2} + \frac{\omega(r-r_0)x}{r}, \quad w = \frac{c(r-r_0)}{r} \tag{37}$$

The stream line shown in Figure 6 while lying on a toroidal surface exhibits a change in the direction of rotation as it traverses the surface because it is forced to use the helical *f* surfaces around the outside and then through the inside of the torus.

Figure 6. Stream surfaces and lines for a torus using the *f* function (36), for ω=37, c=0.9.

A field that more correctly follows behaviour of the line in Figure 1 can be generated by using

$$f = \omega \tan^{-1}\left(\frac{r-r_0}{z-z_0}\right) - c\theta \tag{38}$$

with resulting velocity

$$u = -\frac{c(z-z_0)x}{r^2} + \frac{\omega y}{r}$$
$$v = -\frac{c(z-z_0)y}{r^2} + \frac{\omega x}{r} \qquad w = \frac{c(r-r_0)}{r} \tag{39}$$

and the corresponding stream line (Figure 7), demonstrates that the structure of Figure 1 has been imitated.

Figure 7. Stream surfaces and lines for a torus using the *f* function (38), for ω=20, c=2.

CONCLUSIONS

Dual stream functions which describe certain recirculating flows have been described. In each, one of the stream functions has been used to generate the geometric structure of the flow. The other function then determines the kinematics of the flow within that surface. A more general theory based on this distinction between the two functions is currently being developed.

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