13th Australasian Fluid Mechanics Conference
Monash University, Melbourne, Australia
13-18 December 1998

MODELLING THE MOTION OF AN INTERFACE PAST A THIN PROBE FORMATION OF CONTACT LINES

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ABSTRACT

The interaction of a liquid drop with a thin solid probe facing an oncoming two-phase liquid flow is discussed. It is shown that over much of the flow field inertia and surface tension are the dominant forces acting, though gravity must be accounted for and important boundary layers exist near the surface of the probe (and near the interface).

A simple model is given for determining the steady rise velocity of a drop in a infinite stagnant fluid, which is relevant to this problem.

Lubrication-type equations are developed for the movement of an otherwise planar interface as it wraps symmetrically around a probe having the form of a parabolic cylinder; two regimes are considered, the inviscid-capillary limit and the viscous-capillary limit. Some remarks are made about issues not analysed in this paper.

INTRODUCTION

Recent techniques for sensing the passage of drops and bubbles in multi-phase flow past small (local) probes depend upon understanding the fluid mechanics of the process. Typically, the cross-section of the probe is smaller than that of the drop and it is assumed that the presence of the probe causes little disturbance to the movement of the drop, which appears to remain roughly spherical; however interpretation of the probe signal depends on the fine detail of the way in which the interface between drop and continuous phase approaches the nose (leading point) of the probe and spreads around it until it breaks to form a moving contact line.

We shall consider here the case of one Newtonian fluid, a light oil say, of density ρ_1 and viscisity μ_1, dispersed as drops of diameter D in a second fluid, water say, with which it is immiscible, of density ρ_2 and viscisity μ_2. The interface between the two fluids is characterised by a constant interfacial tension σ. The probe, whose design will be determined partly by its manner of distinguishing between the two fluids, may have a variety of shapes, some of which are shown diagrammatically in figure 1 (a); its characteristic dimension is $d << D$. The axis of the probe is mounted to face (parallel to) the direction of the oncoming flow of dispersed droplets. Its active part will in general be its tip which senses the passage of an interface.

A general flow field will be characterised by $\mathbf{u}(\mathbf{r},t)$, where $\mathbf{u}$ is the velocity of whichever phase (1 or 2) happens to be at position $\mathbf{r}$ at time t. It is possible to write down the mass and momentum equations for such a flow in standard form (Navier-Stokes, no-slip at solid boundaries and Young-Laplace at interfaces) assuming the liquids to be incompressible and isothermal, and drops to remain distinct. For two-phase pipe flow, a typical situation of interest, the full motion will be highly unsteady, particularly if the droplet phase is at all concentrated. However, if the relevant probe dimension, say the radius of curvature of its nose, or the length of the active sensing portion, is small enough, the passage of a drop can be considered to be determined by that of two locally plane interfaces (separating fluids 1 and 2) moving at constant velocity "far" from the probe tip. Thus to each drop will correspond a local size and velocity.

It is observed that sharp tips "puncture" the interface during the passage of any drop that strikes the probe at other than glancing incidence. The time taken for the interface to reach the probe and for the fluid to be squeezed out from the thinning layer between interface and probe tip - see figure 1(b) - is important when interpreting probe signals to yield relative concentration of the two phases and their flow rates.

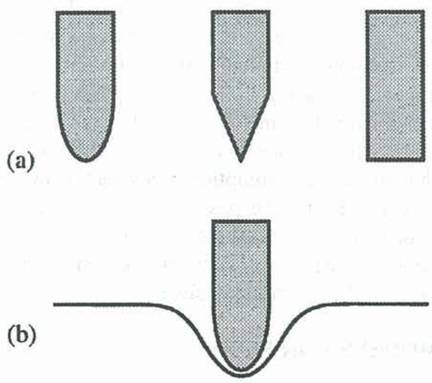

Figure 1 : (a) Simplified diagrams of probe shapes; (b) interface approaching a smooth-nosed probe.

Some aspects of the problem are examined in the following sections. Dimensional analysis is used to estimate the importance of various components of the stress field. A simple mechanical model is given for the buoyant rise of a drop. A convenient probe shape (parabolic cylinder) is selected to illustrate the various stages and regions of the flow process; a plane interface normal to the probe axis moving parallel to the axis, with equal densities and viscosities in the two fluids, is considered for analytic simplicity.

CHARACTERISTIC SCALES; DIMENSIONLESS GROUPS

Typical values of physical quantities are (U_1 and U_2 being mean axial speeds and R a pipe radius):

ρ_1	ρ_2	μ_1	μ_2	σ	U_1	U_1-U_2	D	d	R
kg/m^3	kg/m^3	Pa.s	Pa.s	Pa.m	m/s	m/s	mm	mm	m
750	1000	2	1	0.03	0.4	0.15	5	0.3	0.1

From these we can define and estimate the values of characteristic dimensionless groups that measure the relative significance of the various forces acting:

Reynolds number, drop	$Re_1 = \rho_2(\Delta U)D/\mu_2 \approx 750$
Reynolds number, probe	$Re_1 = \rho_2 U_1 d/\mu_2 \approx 150$
Capillary number	$Ca = \mu_2 U_2/\sigma \approx 0.01$
Weber number, drop	$We_1 = \Delta\rho(\Delta U)^2 D/\sigma \approx 1$
Weber number, probe	$We_2 = \rho_2 U_1^2 d/\sigma \approx 1$
Bond number, drop	$Bd_1 = \Delta\rho g D^2/\sigma \approx 2$
Bond number, probe	$Bd_1 = \Delta\rho g d^2/\sigma \approx 0.01$

From these, we can confirm that capillarity and inertia are the dominant forces acting almost everywhere. The main effect of gravity is to distort the larger drops from the spherical and will be insignificant for probe-interface dynamics. Viscosity will lead to thin boundary layers over the probe (of the order of 10 μm) and to weak effects near moving interfaces.

The pipe Reynolds number $Re_p = \rho_2 U_2 R/\mu_2 \approx 40{,}000$ so we expect turbulent flow. An interesting question is whether the turbulent eddies are such as to cause droplet distortion or unsteadiness in the incoming flow during the passage of a drop interface past a probe tip. The time of passage is of the order of $d/U_1 \approx 1$ ms. By using the Kolmogorov scale (Batchelor, 1953, p.115) for eddy length $(\nu^3/\varepsilon)^{1/4}$ and velocity $(\nu\varepsilon)^{1/4}$ where ε is the local rate of isotropic turbulent dissipation/unit mass, we obtain typical values of 100μm and 1 cm/s respectively, with a characteristic time of 10 ms. We conclude that the turbulent fluctuations will only cause small perturbations to the assumption of a steady flow far from the probe tip during the passage of an interface. More detail about pipe flow turbulence can be obtained from Townsend (1956); more elaborate reasoning based on his data leads to a similar conclusion.

SUBMODELS FOR FLOW

Bubble rise velocity

A surprisingly useful estimate (to compare with observations) can be obtained by supposing the flow field for a rising drop to be given by a Hill spherical vortex (Batchelor, 1967, p.526) moving at speed ΔU relative to the fluid at rest far from the vortex. This flow field, steady with respect to the centre of the vortex, is a solution of the inviscid equations of motion. Within the vortex, the fluid has uniform vorticity[1]; outside the vortex it is irrotational. Furthermore the velocity is continuous at the spherical surface defining the vortex. For the $O(10^3)$ Reynolds numbers characterising rising drops, an inertia-dominated flow field is to be expected; for relatively small differences in density, drop Bond and Weber numbers will be small enough to ensure near sphericity, as is observed. The balance of buoyancy and drag can be understood by supposing the viscous dissipation to be balanced by the loss of gravitational potential energy, as has been done successfully for gas bubbles within an appropriate volume range (see Batchelor, 1967, fig. 5.14.1). For a viscosity ratio $m = \mu_1/\mu_2$ the dissipation within the drop can be shown to be $4m$ times that outside. For our case, estimates for rise speeds within a factor of 2 of those observed were obtained.

Flow past a parabolic cylinder

This specific probe shape will be used here for illustrative purposes. Flows will be taken as plane and both $r = \rho_1/\rho_2$ and m to be 1 for simplicity. The probe surface cross-section is given by

$$y^2 = 4c(c-x) \qquad (1)$$

where $y = 0$ is the probe axis (plane of symmetry) and its nose is at $(c,0)$. It extends from c to $-\infty$. Its curvature is

$$\kappa = -c^{1/2}/2(2c-x)^{3/2} \qquad (2)$$

with a maximum $-1/2c$ at the nose. It is convenient to use parabolic cylinder co-ordinates (see Happel and Brenner, 1965, p.500)

$$x = c(\xi^2 - \eta^2),\ y = 2c\xi\eta, \qquad h_\xi = h_\eta = 1/2c(\xi^2+\eta^2)^{1/2} \qquad (3)$$

in terms of which the probe surface is $\xi = 1$.
Inviscid single-phase flow is given by the stream function

$$\psi = 2Uc(\xi - 1)\eta \qquad (4)$$

where

$$u_\eta = h_\xi \frac{\partial\psi}{\partial\xi},\quad u_\xi = h_\eta \frac{\partial\psi}{\partial\eta} \qquad (5)$$

and $u_x = -U$ is the uniform flow at $x \to \infty$ ($\xi \to \infty$).

If the surface tension σ were negligible ($We_2 \to \infty$) then an interface would simply be convected. It can be seen that along the axis $y = 0$, the fluid speed will be $U(\xi^{-1} - 1)$, and so the time taken to move the interface from $(x_1,0)$ to $(x_2,0)$ will be given by

$$t_{12} = \int_{x_1}^{x_2} \frac{dx}{u_x(x,0)} = \int_{(x_2/c)^{1/2}-1}^{(x_1/c)^{1/2}-1} \frac{2c(1+\zeta)^2}{U\zeta} d\zeta \qquad (6)$$

For $y \to \infty$, the equivalent time would be simply $(x_1 - x_2)/U$. The difference Δt represents the slowing up effect of the probe and yields the drift

$$U\Delta t = 2c^{1/2}(x_1^{1/2} - x_2^{1/2}) + 2c\ln\frac{x_1^{1/2} - c^{1/2}}{x_2^{1/2} - c^{1/2}} \qquad (7)$$

Note that this becomes unbounded as $x_1 \to \infty$ (although the flat interface assumption becomes invalid as $x_1 \to D$)

[1] Surprisingly this inner flow field is the same as that obtained (Batchelor, 1967, p.235) in the low Reynolds number limit, the Hadamard solution

but also as $x_2 \to c$: the interface never reaches the surface of the probe, because there is a stagnation point at the nose even in the single-phase inviscid solution.

If the surface tension is dominant ($We_2 \to 0$) the interface is convected as a line (plane) until it hits the nose, taking a finite time from any finite position, say $x_1 = D$.

If we use the local velocity $|u_x|$ and distance from the nose $(x - c)$ to estimate local Weber and Reynolds numbers, we find that both decrease towards zero, and so as expected viscosity becomes important and surface tension provides the driving force.

The full single-phase boundary-layer solution satisfying no slip at the probe surface can be obtained for the outer flow (4) (see Haddad and Corke, 1998), the parabolic cylinder co-ordinates (ξ,η) being optimal in the Kaplun sense (Rosenhead, 1963, p.240). For our purposes we note that very near the nose, along $\eta = 0$, the solution is essentially that for the flat plate stagnation point solution (Rosenhead, 1963, p.155) with outer flow

$$u_{x'} = -Ux'/c, \quad u_{y'} = Uy'/c \tag{8}$$

and so the effective boundary layer thickness $l(0)$ will be about $2(\mu c/\rho U)^{1/2}$. For our typical case we take $c = 100$ μm, giving $l(0) \approx 30$ μm, which is suitably small compared to the maximum radius of curvature $2c = 200$ μm.

Lubrication approximation

As the interface approaches the nose, it will wrap around it so that a thin "lubricating" layer of the fluid being displaced separates the interface from the surface of the probe. A suitable criterion for this is that the thickness $h(\eta)$ be small compared to the inverse of the curvature $\kappa(\eta)$. From (2) and (3) above, putting $\xi = 1$ on the probe, this requires

$$h(\eta) << 2c(1+\eta^2)^{3/2} \tag{9}$$

We note that (9) is not incompatible with

$$l(\eta) << h(\eta) \tag{10}$$

for extreme values of the parameters characterising the flow, and so an inviscid lubrication approximation is not without interest. For this situation $u_\eta(\eta,\xi) \approx \underline{u}(\eta)$, constant across the layer while the flow outside the layer is the single-phase inviscid solution (4) to the same order of approximation. Because the layer is thin, the curvature of the interface will be that of the probe itself, while the outer flow pressure field at the interface will be that at the probe in the absence of an interface. This means that the pressure $\underline{p}$ within the layer can be written

$$\underline{p}(\eta) = -\frac{\rho U^2 \eta^2}{2(1+\eta^2)} + \frac{\sigma}{2c(1+\eta^2)^{3/2}} \tag{11}$$

The equations governing the flow are

$$\frac{\partial \underline{u}}{\partial t} + \frac{\underline{u}}{2c(1+\eta^2)^{1/2}}\frac{\partial \underline{u}}{\partial \eta} = -\frac{1}{2c\rho(1+\eta^2)}\frac{\partial \underline{p}}{\partial \eta} \tag{12}$$

and $$\frac{\partial h}{\partial t} + \frac{1}{2c(1+\eta^2)^{1/2}}\frac{\partial (h\underline{u})}{\partial \eta} = 0 \tag{13}$$

the first being the unsteady Euler equation and the second the integrated form of the continuity equation. Because the driving force in (12), namely $\underline{p}$ given by (11), is independent of time, we can put $\partial \underline{u}/\partial t \approx 0$ for late times, and so (11), (12) become decoupled from (13). We obtain

$$\frac{1}{2}\frac{\partial \underline{u}^2}{\partial \eta} = \frac{U^2\eta}{(1+\eta^2)^2} + \frac{3\sigma}{2\rho c}\frac{\eta}{(1+\eta^2)^{5/2}} \tag{14}$$

which can be integrated, using the boundary condition $\underline{u}(0) = 0$, to give

$$\underline{u}^2 = U^2\left(1-\frac{1}{1+\eta^2}\right) + \frac{\sigma}{\rho c}\left(1-\frac{1}{(1+\eta^2)^{3/2}}\right) \tag{15}$$

For $\eta \to \infty$, we note that

$$\underline{u} \to U\left(1+\frac{\sigma}{\rho c U^2}\right)^{1/2} \tag{16}$$

The estimated value for We_2 shows that neither term in (16) can be expected to dominate. The solution of (13) using (15) is best carried out computationally, but we can consider the situation at $\eta = 0$, where (13) simplifies to

$$\frac{\partial h_0}{\partial t} + \frac{Uh_0}{2c}\left(1+\frac{3\sigma}{2\rho c U^2}\right)^{1/2} = 0 \tag{17}$$

and so h_0 decreases exponentially with a time scale of the order of c/U as expected. Similar exponential decrease can be expected at other values of η.

However this process will only be relevant until viscosity becomes important. For the final phase of approach of the interface, when $h(0,t) << l(0)$, a purely viscous lubrication approximation within a thin layer with a no-slip condition at the probe surface, and a no-tangential-stress condition at the interface, will be relevant.It will be driven by the same pressure field (11) as obtained for the inviscid case. The local flow equations become

$$\frac{1}{2c(1+\eta^2)^{1/2}}\frac{\partial \underline{p}}{\partial \eta} = \mu\frac{\partial^2 u_\eta}{\partial n^2} \tag{18}$$

where n is a local co-ordinate normal to the interface (and the probe surface), along with (13) where

$$h\underline{u} = \int_0^h u_\eta(n)dn \tag{19}$$

Using the no-slip and no-stress boundary conditions on u_η at $n = 0$ and h yields

$$h\underline{u} = -\frac{h^3}{6\mu c(1+\eta^2)^{1/2}}\frac{\partial \underline{p}}{\partial \eta} \tag{20}$$

and so

$$\frac{\partial h}{\partial t} + \frac{\rho U^2}{12\mu c^2(1+\eta^2)^{1/2}}\frac{\partial}{\partial \eta}\left(\frac{h^3\eta}{(1+\eta^2)^{5/2}} + \frac{3\sigma\eta/2\rho c U^2}{(1+\eta^2)^3}\right) = 0 \tag{21}$$

Once again we consider the situation at $\eta = 0$, to obtain

$$\frac{\partial h_0}{\partial t} = -\frac{\rho U^2}{12\mu c^2}\left(1+\frac{3\sigma}{2\rho c U^2}\right)h_0^{\,3} \qquad (22)$$

with solution $h_0 \propto t^{-1/2}$.

The intermediate "boundary-layer" phase, when both inertia and viscosity are relevant on both sides of the interface is a more difficult problem to set out, though of considerable mathematical interest. The interfacial boundary conditions will now require continuity of tangential shear stress and velocity; the pressure field will be given by both terms in (11) within the layer and the inviscid solution outside. The scale length for the normal co-ordinate will be l and for the tangential co-ordinate c. Both Weber and Reynolds numbers will play a part in the evolution of h_0 with t. The nature of any important similarity solutions has not been fully investigated.

DISCUSSION

The aim of this work was not to give definitive solutions to an idealised problem. It was to obtain insight into a technically important flow and to identify the most important issues worth investigating in a more comprehensive study. There are several related issues that have been considered but have not been reported upon here.

(i) Boundary integral methods for calculating the motion of an inviscid drop in an inviscid fluid flow approaching a probe, where the problem length scale is several drop diameters and the full shape of probe and housing are considered. Bond and Weber numbers for the drop suggest that both gravity and surface tension need to be included to give realistic results, while the vorticity within the drop has to form part of the initial conditions.

(ii) Non-symmetric drop arrival at the probe, corresponding, in the near-probe flow approximation considered above, to an inclined interface and a non-axial uniform far flow field; in practice both of these could be treated as perturbations to the symmetric flow field, in that most drop/probe encounters for $D >> d$ (which contribute significantly to estimates for fractional concentration) will be covered by small departures from normal incidence of the drop surface at the probe tip.

(iii) Sharp-tipped, sharp-edged and axisymmetric probes; a two-dimensional probe shape and planar flow field were considered here for simplicity and an equivalent analysis for a paraboloidal probe and an axisymmetric flow field has been carried out, leading to similar results and conclusions; however wedge and cone shapes for the probe do not lead to simple uniformly valid outer flow fields - the relevant singularities at the tip are not easily accommodated within boundary-layer theory (even the limit $c \to 0$ giving a flat plate in the parabolic cylinder case causes problems). From an operational point of view, tips and edges are important in that they precipitate a necessary breaking of the interface allowing the drop fluid to displace the continuous fluid and vice-versa; it is well known in Stokes flow solutions that sharp bodies can touch one another whereas smooth spheres can never touch when pushed together by finite forces.

(iv) Contact lines moving over the probe surface; aspects of this problem have been considered by many authors, but the full process involved here, including the formation of a contact line, is not a matter for traditional fluid mechanics alone; physico-chemical aspects of wetting play a significant part as experiment has shown and are critical at the technical level; a satisfactory set of conservation and constitutive equations leading to well-posed mathematical problems has not yet been agreed upon.

Experiments will continue to play an important part in future investigations, but it should be noted that optical means are limited in resolution by the wavelength of light ($\approx$ 1 μm) which is just insufficiently small to resolve aspects of contact line behaviour. It seems likely that comparison of electrical and optical probe observations with theoretical solutions for interface movement will be the best way of gaining a full understanding of the processes involved.

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