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SILICIC LAVA DOMES ON SLOPES

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ABSTRACT

A theoretical and experimental analysis is made of a static, isothermal Bingham fluid lying on an inclined plane. This analysis is aimed at understanding the shapes of highly siliceous lava flows which occur on steep mountain sides or on gentle slopes. It is thought that such flows have a finite yield strength, and we investigate the static balance which governs the final shape of the lava dome.

The results indicate that there is a limit to the vertical thickness of a lava dome on a slope, depending on the yield strength of the fluid and the underlying slope. For domes with heights less than half this limiting value, the shape of the dome is close to axisymmetric. Domes with heights approaching this limit become highly elongated in the downhill direction.

INTRODUCTION

Lava domes form through the effusion of magma from a vent. It has long been known that for these domes, the flow displays significant non-Newtonian behaviour. Indeed, were they Newtonian fluids, they would continue to flow, albeit possibly extremely slowly, until they pool in a depression. Instead, these lava domes undergo only restricted spreading. The deviations from Newtonian behaviour can in some instances be primarily explained through the formation of a crust with a finite yield strength, but it has also been suspected that for the highly silicic lavas, a finite, internal yield strength also restricts the spreading, even before significant cooling occurs.

Hulme (1974) estimated that for lava flows greater than a few metres thick, the crust that forms on its exterior as it cools would not be strong enough to significantly restrict the flow. Assuming the lava flows were Bingham fluids, he formulated equations to relate the flow rate to the ratio of the basal shear stress to the yield strength, a parameter that could be deduced from the geometry of the flow. His theories were thus applicable to lava flows that had features characteristic to viscous flow of a Bingham fluid, quite distinct to a Bingham dome in static balance.

Blake (1990) derived a formula for the growth of such a Bingham dome on a flat surface, using a radial depth profile first derived by Nye (1952) in the context of glaciers. This shape was also used by Hulme, however Blake was only interested in very slowly growing domes, where the spreading was governed solely by yield strength, not by viscous forces. Griffiths and Fink (1997) extended this work of Blake by including the effects of cooling at the surface, and thus the formation of a crust. They were able to produce a scaling analysis from which it is possible to predict conditions under which the cooling is not rapid enough to significantly affect the shape of the lava dome.

Here we aim to further Blake's work on isothermal Bingham fluids by finding solutions for the shapes of flows on sloping surfaces, again for the case of very slow extrusions and the final static dome, where viscous forces are negligible.

THEORY

Let us consider a hydrostatic, isothermal blob of lava on a gently sloping planar base. We assume a uniform internal yield strength and neglect effects of surface tension. When the flow is sufficiently slow or has come to rest, the stress within the fluid at the very base due the internal pressure gradient is balanced exactly by its internal yield strength. Referring to the geometry in Figure 1, the force balance is given by

$$(g\Delta\rho)^2\left[\left(\frac{\partial h^*}{\partial x^*}-\sin\alpha\right)^2+\left(\frac{\partial h*}{\partial y^*}\right)^2\right]=\left(\frac{\sigma}{h^*\cos\alpha}\right)^2, \quad (1)$$

where g is gravity, $\Delta\rho$, h^* and σ are the anomalous density, vertical thickness and internal yield strength of the lava, and α is the slope of the ground from horizontal, presumed to be small. Stars indicate dimensional lengths. We have also assumed that the flow is wide and thin so that the stress gradient is greatest in the z direction, an approximation which tends to break down near the edge of the dome where the surface is almost perpendicular.

Let us nondimensionalise the horizontal length scales by the quantity $\sigma/(g\Delta\rho\sin^2\alpha\cos\alpha)$ and the verti-

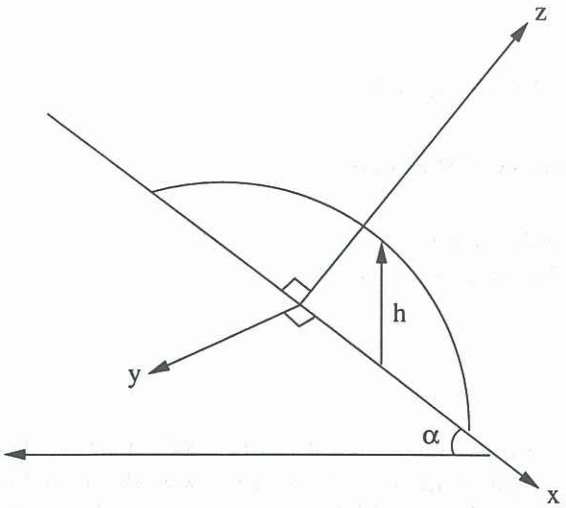

Figure 1: Geometry of the dome. The x axis is aligned parallel to the sloping ground directed downhill, the y axis is directed across the slope, while the z axis is perpendicular to it. The thickness h is measured vertically, not in general parallel to the z axis.

cal length scale by the length $\sigma/(g\Delta\rho \sin\alpha\cos\alpha)$. Note that these scales do not generally correspond to actual lengths as they tend to infinity as the slope tends to zero, whereas the physical length scale of the lava blob will tend to infinity in the down-slope direction as the slope goes to 90 degrees. Rather, they indicate the extent to which gravity pulls the dome downhill. If the dimensionless length scales are small, then the lava dome spreads almost axisymmetrically on the slope, while large scales indicate the lava dome spreads mainly down the slope. With these scales (1) becomes

$$\left(\frac{\partial h}{\partial x} - 1\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 = \left(\frac{1}{h}\right)^2, \quad (2)$$

where all length scales are now dimensionless.
On the x axis, symmetry arguments make $\partial h/\partial y = 0$ Thus (2) becomes

$$\left(\frac{\partial h}{\partial x} - 1\right)^2 = \left(\frac{1}{h}\right)^2, \quad (3)$$

which has the solution on the down-slope side as

$$x = h - H + \ln\left|\frac{1-h}{1-H}\right|, \quad (4)$$

where $h = H$ when $x = 0$, while on the up-slope side the solution is

$$-x = H - h + \ln\left|\frac{1+h}{1+H}\right|. \quad (5)$$

The up-slope and down-slope extent of the flow may be found by substituting $h = 0$ into (4) and (5), giving

$$x_d = -H - ln|1 - H|, \quad (6)$$

$$-x_u = H - ln|1 + H| \quad (7)$$

or a total length of

$$L = -ln|1 - H^2|. \quad (8)$$

There are a couple points to notice about this. First of all, when the dimensionless thickness H tends to 1, the length of the lava dome on the down-slope side, x_d, goes to infinity. Thus we have a condition on the critical thickness of the lava flow at which the yield strength of the fluid is unable to support the dome on the slope in a static balance with gravity. In dimensional units that condition becomes:

$$H^* = \frac{\sigma}{g\Delta\rho \sin\alpha\cos\alpha}. \quad (9)$$

We shall only concern ourselves with lava domes that can be supported in the static balance, so from now on we shall assume $H < 1$.
The second point of note is that our length scales tend to infinity and thus our dimensionless quantities tend to zero when the slope approaches zero. This case also applies when the lava has a very large internal yield strength and when the thickness (ie volume) of the lava dome is extremely small. For small values of any variable u, $\ln(1 + u) \approx u - u^2/2$, so that equations (4) and (5) reduce to the equation:

$$|x| \approx \frac{1}{2}(H^2 - h^2), \quad (10)$$

or in dimensional quantities

$$|x^*| \approx \frac{g\Delta\rho\cos\alpha}{2\sigma}(H^{*2} - h^{*2}). \quad (11)$$

This thickness profile is consistent with the formula derived by Nye (1952) for the cross-sectional shape of a glacier on level ground, using identical assumptions to our own.
The cross-sectional shape of the dome along its axis of symmetry, given by equations (4) and (5) is plotted for several values of H in Figure 2. Notice how the dome becomes increasingly asymmetric as the central thickness, H, is increased. Perhaps the most striking aspect of Figure 2 is the discontinuity in the slope of the cross-sectional shape at the origin. This is due to the assumption that the internal yield strength is exactly balanced by the gravitational stresses (due to surface slope) at every position. Such an outcome might be reached through only certain flow histories, in particular, after extremely slow extrusions of the fluid from a source at the origin. In that case, the lava dome will have grown outwards from the origin, so that the internal yield strength of the lava is directed to oppose the flow produced by the increasing

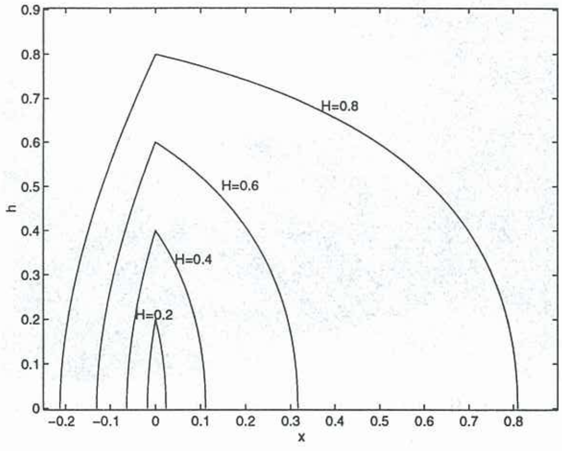

Figure 2: Dimensionless cross-sectional shape of the lava dome along its axis of symmetry, as a function of the thickness of the dome

volume. That is, for negative x values, on the up-slope side, the yield strength acts in a down-slope direction, while for positive x the yield strength is acting in a up-slope direction.

While it may appear worrisome that the shape of the lava dome is dependent on the past history of its shape, we should realise that on the down-slope side of the origin, one would not expect the dome to have ever flowed upstream. Thus the assumption that the internal yield strength is applying a force directed uphill is quite safe, and so we can at least be confident of the shape of the dome down-slope of the origin, which is really the side in which we are primarily interested.

Another length scale one may get from (2) is if we assume $\partial h/\partial x << 1$. We would expect this to be true along $x = 0$, especially when the lava dome is reasonably axisymmetric ($H << 1$). In this instance equation (2) reduces to

$$1 + \left(\frac{\partial h}{\partial y}\right)^2 = \left(\frac{1}{h}\right)^2 \qquad (12)$$

which has the solution

$$y = \pm(\sqrt{1-h^2} - \sqrt{1-H^2}). \qquad (13)$$

Setting $h = 0$ into (13) gives us an approximation to the across-slope width of the dome

$$W \approx 2(1 - \sqrt{1-H^2}). \qquad (14)$$

A plot of the width of the dome compared to its length is given in Figure 3. One can think of this as being the plot of width versus length of a lava dome as it grows with time, so long as the viscous forces within the dome are never large enough to be significant.

We now have expressions for the cross-sectional shape of the dome along its axis of symmetry, and for its

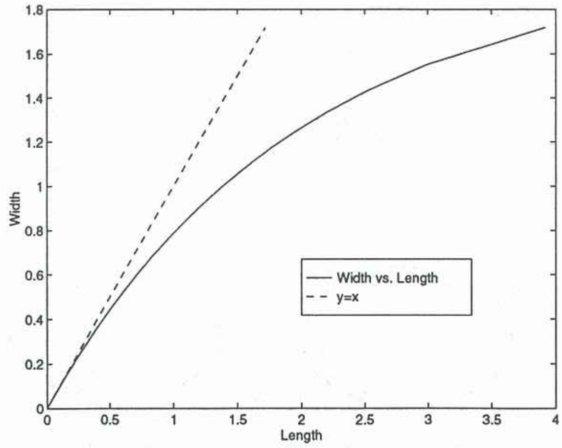

Figure 3: Width verses length of the lava dome.

width. With these we may deduce a scaling law for the volume:

$$V \sim \cos\alpha\,(1 - \sqrt{1-H^2})\left(\ln\left|\frac{1+H}{1-H}\right| - 2H\right), \qquad (15)$$

where the $\cos\alpha$ term arises due to the fact that h is not measured perpendicular to the slope.

For the case of small H, and when $\alpha = 0$

$$\lim_{H\to 0} V = \frac{H^5}{3}, \qquad (16)$$

which again is in agreement with the volume derived from Nye's glacial work: $V = \frac{2\pi}{15}H^5$.

NUMERICAL SOLUTIONS

Equations (4) and (5) allow us to find a solution to (2) by treating it as an initial value problem. We rewrite (2) as

$$\frac{\partial h}{\partial y} = -\left[\left(\frac{1}{h}\right)^2 - \left(\frac{\partial h}{\partial x} - 1\right)^2\right]^{1/2}, \qquad (17)$$

where we have assumed that the thickness of the dome reduces as one moves away from its axis of symmetry plane. On a discrete grid this becomes

$$\begin{aligned} h_{i,j+1} &= \tfrac{1}{2}(h_{i+1,j} + h_{i-1,j}) \\ &\quad - \Delta y\left[\left(\tfrac{1}{h_{i,j}}\right)^2 - \left(\tfrac{h_{i+1,j}-h_{i-1,j}}{2\Delta x} - 1\right)^2\right]^{1/2} \end{aligned} \qquad (18)$$

where the Lax method has been employed (using $(h_{i+1,j} + h_{i-1,j})/2$ instead of $h_{i,j}$ for the first term on the right hand side of the equation) to increase stability in the algorithm. If we begin with the initial condition given by (4) and (5), and then integrate over the width of the dome using (18), we can obtain the shape of the lava dome. This has been done in Figure 4 for various dimensionless values of H shown.

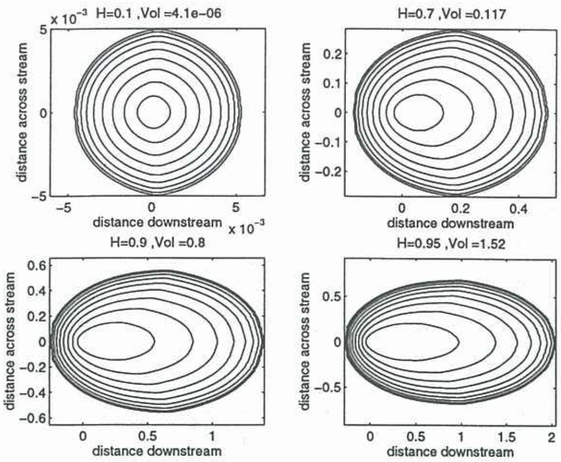

Figure 4: Contour plots of the numerical solution to equation 2. Contours show equal intervals in thickness from 0 to H

The numerical solutions also provide a value for the volume of each lava dome (actually $V/\cos\alpha$). These are displayed above each figure, to the accuracy justified by investigating lower resolution solutions. These volumes are then plotted against our scaling result given by (15) in Figure 5, which indicates a very good relationship between theory and the numerical result. The numerical solutions converge less rapidly for the runs with larger values of H, and thus have higher levels of uncertainty in the quoted volumes, however even those error estimates are too small to be visible in Figure 5.

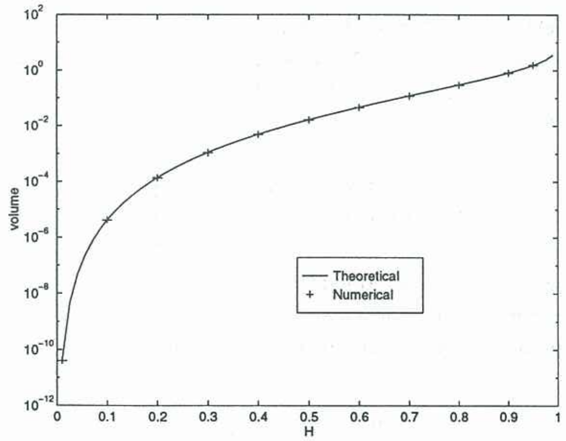

Figure 5: Comparison between the numerical and theoretical volumes of the lava domes as a function of their dimensionless thickness.

EXPERIMENTAL METHOD

Experiments designed to test the theoretical results are currently underway. Various mixtures of Polyethylene Glycol (PEG) wax and Kaolin, and water and Kaolin were used to create the Bingham fluid. Known quantities of the slurry are injected at room temperature very slowly through a hole in the base of a tank. When all motions in the fluid have ceased, the horizontal dimensions of the flow are measured. For each mixture, the first run is done with the base of the tank horizontal, and then the yield strength of the fluid is measured using the dimensional form of (16), or

$$\sigma = \frac{225V^2 g\Delta\rho}{64\pi^2 C^2 r^5}, \tag{19}$$

where $C = 1.76$ is Blake's empirically derived version of the proportionality constant, compared to $C = \sqrt{2}$ given by theory.

With the tank inclined at various angles, a small amount of the slurry is injected through the base. The dimensionless width and length are measured, then some more slurry is injected and another measurement made. This procedure can be repeated until the the blob reaches a wall of the tank. Figure 6 is a photograph of one of the runs at a particular volume. The experimental results will be reported in greater detail at the conference.

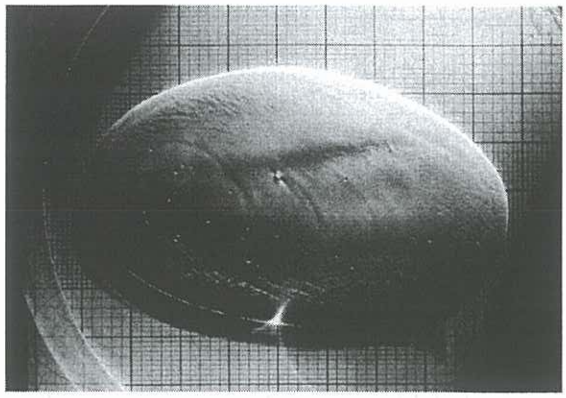

Figure 6: Photograph of one of the experimental runs, at a particular volume.

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