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ZERO DIVERGENCE FINITE ELEMENT VISCOUS INCOMPRESSIBLE FLOW SIMULATION

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ABSTRACT

The solenoidal finite element approach is a novel idea for the solution of viscous incompressible flows. Its name derives from the fact that, due to the need to conserve mass, the velocity components must have zero divergence, a constraint which is imposed at the element level. The main difficulty with this approach is the construction of a specialised element in which the velocity components are constrained to be solenoidal by the nature of their interpolation functions. A feature of the approach is the suppression of the pressure from the prime solution and its subsequent retrieval in the manner of an auxiliary variable. The validity of the approach is demonstrated by the results for a classical benchmark problem. Internal consistency is proved by comparisons between solutions from different meshes. External consistency is proved by comparisons with solutions, both from other codes and from the literature.

INTRODUCTION

An idea which has received scant attention in connection with finite element solutions to viscous incompressible flows is the solenoidal approach. The attraction of this approach is its inherent satisfaction of the continuity constraint and its uncoupling of the pressure from the prime solution. Early reviews (Tuann and Olson, 1978; Norrie and de Vries, 1978) of finite element solution methods commented on the difficulty, if not impossibility, of the approach. Nevertheless, after numerous setbacks, success was achieved (Mack, 1994). In the process, the three main primitive variable approaches (solenoidal, penalty function, Lagrange multiplier) were unified and shown to be but different manifestations of the imposition of the continuity constraint (Mack, 1984).

The difficulty with the solenoidal approach centres on the construction of an admissible element, one which exhibits zero divergence. To date, such elements have proved somewhat elusive. The element which is employed here is a development of the first genuine solenoidal element (Mack, 1990).

The solenoidal approach suppresses the pressure from the prime solution. The velocity components are determined solely from their own values at the previous iteration. Once the velocity components have been found, the pressure can be retrieved whenever it is desired, as though it was an auxiliary variable. Details are provided herein of the mathematical process which achieves this.

INTEGRAL FORMULATION

Consider the steady plane laminar flow of an incompressible Newtonian fluid whose viscosity is constant. The governing equations for this case are

$$u_x + v_y = 0\,, \tag{1}$$

$$\frac{1}{Re}\nabla^2 u - p_x = uu_x + vu_y\,, \tag{2.1}$$

$$\frac{1}{Re}\nabla^2 v - p_y = uv_x + vv_y\,, \tag{2.2}$$

where p is the pressure with respect to some datum, u is the velocity component in the x direction, v is the velocity component in the y direction, all of which have been normalised, in doing so introducing the Reynolds number Re.

The finite element method operates, not on the differential equations, but on an integral formulation. Such a formulation can be obtained from an inner product with the arbitrary variations δp, δu, δv so that

$$\int_A \delta p\left(u_x + v_y\right) dA + \int_A \delta u\left(\frac{1}{\mathrm{Re}}\nabla^2 u - p_x - uu_x - vu_y\right) dA + \int_A \delta v\left(\frac{1}{\mathrm{Re}}\nabla^2 v - p_y - uv_x - vv_y\right) dA = 0\,, \tag{3}$$

where A is the integration domain. With the utilisation of Green's theorem, this becomes

† Part of the research for this paper was undertaken whilst the author was on leave at the Institute for Numerical Methods in Engineering, University of Wales, Swansea, United Kingdom.

$$\frac{1}{\text{Re}}\int_A \left(\delta u_x u_x + \delta v_x v_x + \delta u_y u_y + \delta v_y v_y\right) dA$$
$$+\int_A \left[\delta u\left(u u_x + v u_y\right) + \delta v\left(u v_x + v v_y\right)\right] dA$$
$$= \int_A \left[\delta\left(u_x + v_y\right) p + \delta p\left(u_x + v_y\right)\right] dA$$
$$+\int_s \left[\delta u\left(\frac{1}{\text{Re}} u_n - \alpha p\right) + \delta v\left(\frac{1}{\text{Re}} v_n - \beta p\right)\right] ds, \quad (4)$$

where α, β are the direction cosines for the normal n to the boundary s.

Although often not formulated as such, the main primitive variable approaches all follow this path. However, the solenoidal approach then takes the radical step which recognises the advantage to be gained from the imposition at the element level of velocity components with zero divergence. If this can be achieved, then

$$\frac{1}{\text{Re}}\int_A \left(\delta u_x u_x + \delta v_x v_x + \delta u_y u_y + \delta v_y v_y\right) dA$$
$$+\int_A \left[\delta u\left(u u_x + v u_y\right) + \delta v\left(u v_x + v v_y\right)\right] dA$$
$$= \int_s \left[\delta u\left(\frac{1}{\text{Re}} u_n - \alpha p\right) + \delta v\left(\frac{1}{\text{Re}} v_n - \beta p\right)\right] ds, \quad (5)$$

whereupon the pressure is eliminated from the prime solution. Note, however, that Eq. (5) is dependent on the satisfaction, not of the solenoidal constraint (1), but of the weaker statement

$$\delta \int_A p\left(u_x + v_y\right) dA = 0. \quad (6)$$

As already mentioned, the difficulty with the solenoidal approach centres on the construction of an element for the velocity components. A triangular element is constructed in which Eq. (1) is satisfied at Gauss points so that

$$\int_{A_e} \left(u_x + v_y\right) dA = 0. \quad (7)$$

In this manner, the constraint is imposed both in a collocation sense and in an integral sense. Eq. (7) is, nonetheless, a relaxation of Eq. (1) and an approximation to Eq. (6). Precise satisfaction of Eq. (6), and thus Eq. (5), will occur only for constant pressure, or in the limit as the size of the element tends to zero. Such a relaxation is not uncommon in integral formulations for incompressible flow. Most notably, analogous restrictions are found necessary in the penalty function approach and the Lagrange multiplier approach.

The element employed here is an improved version of that reported in earlier work. One feature is that, rather than use a complete quartic, it uses a complete quadratic with a quartic bubble. This provides the element with the higher order accuracy without the drawback of the associated higher storage requirements. Another feature is the tighter satisfaction of Eq. (7) with double precision.

BENCHMARK TESTS

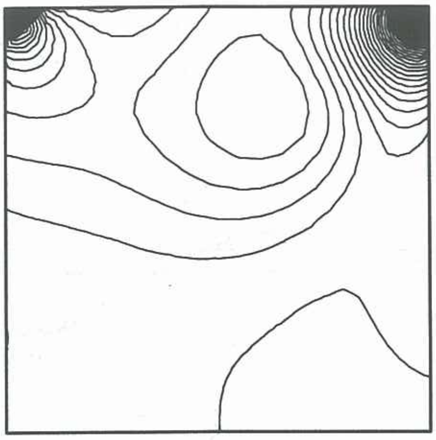
The procedure described here forms the basis of the SOLFEM code. This produces solutions for the velocity components and the pressure, as well as the auxiliary quantities which are derivable from these. The entire code is written from scratch, even to the extent of the graphics. Solution is by means of a frontal scheme. Convergence is considered to occur when the maximum change in nodal values of velocity components between successive iterations is less than 10^{-5}.

The validity of the procedures which are implemented in the SOLFEM code is tested by the application of the code to the classical benchmark problem of the driven flow in a square cavity. This flow, which is maintained by the action of a sliding lid, is a prototype for flows with recirculation.

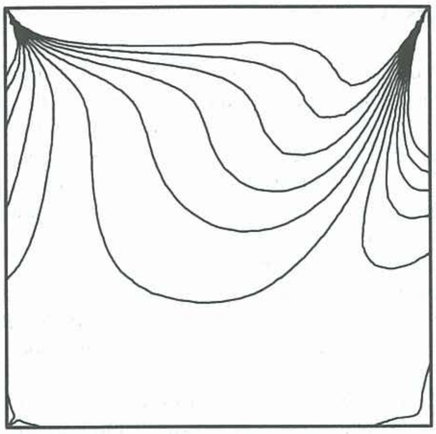
Results from the SOLFEM code are provided here for various Reynolds numbers. A solution at a particular Reynolds number is initiated from the solution at an intermediate Reynolds number. Convergence never takes more than 5 iterations. Meshes are composed of almost uniform triangle pairs. The mesh density m is the number of these pairs along each side. Sample solutions, for m=60 and Re=100, are presented in Figure 1 through Figure 4. These comprise pressure contours, vorticity contours, streamfunction contours, wall pressure coefficient profiles and centre-line velocity profiles. Table 1 details the properties at the vortex core. Where necessary, the value of m is appended to the code name to differentiate between the meshes.

Comparisons are made with the results from the GALERKIN code, based on the standard Galerkin (Lagrange multiplier) approach and written by the author purely as a benchmark for the SOLFEM code. As well, results are included from the commercial codes, FLUENT and FIDAP. The former is a finite volume code for which 66 x 66 cells are used. The latter is a finite element code for which 80 x 80 elements are used. Table 1 also includes results from the reference (Olson and Tuann, 1979), denoted by OT.

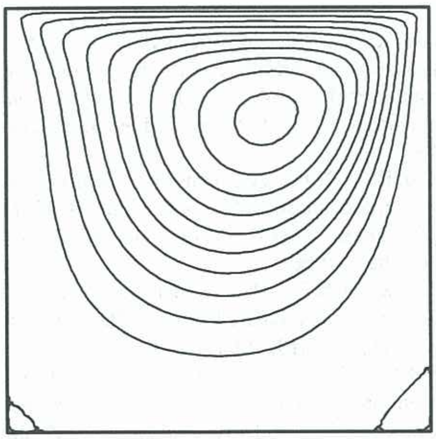
In general, the results are in good agreement. Bear in mind that, for each code, the mesh is, at most, slightly graded and certainly not optimum. Also, each code uses a different order interpolation. Therefore, any difference between the results is likely to be related to a difference in the resolution of the meshes. For the SOLFEM code, Table 1 clearly demonstrates the improvement in the solutions with the refinement of the mesh.



Pressure contours, p=0, +Δp=.02, -Δp=.02

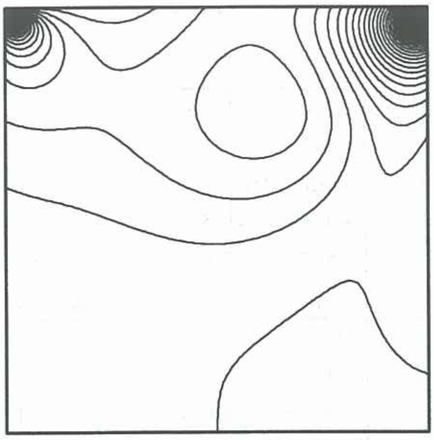


Vorticity contours, ω=0, +Δω=1, -Δω=1

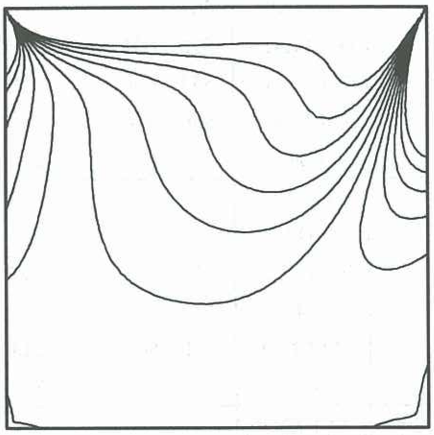


Streamfunction contours, ψ=0, +Δψ=.0000001, -Δψ=.01

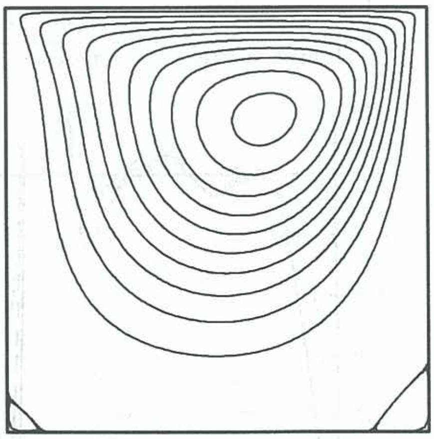
Figure 1: GALERKIN code, Re=100



Pressure contours, p=0, +Δp=.02, -Δp=.02



Vorticity contours, ω=0, +Δω=1, -Δω=1



Streamfunction contours, ψ=0, +Δψ=.0000001, -Δψ=.01

Figure 2: SOLFEM code, Re=100

Re	Code	v.c. properties				
		x	y	p	ω	ψ
0	FLUENT	.500	.765	-	-	-
	FIDAP	.500	.766	-	-	-
	SOLFEM30	.500	.766	-	-3.23	-.100
	SOLFEM60	.500	.766	.000	-3.22	-.100
	GALERKIN	.500	.766	.000	-3.22	-.100
	[OT]	.50	.76	.000	-3.12	-.100
100	FLUENT	.620	.748	-	-	-
	FIDAP	.618	.731	-	-	-
	SOLFEM30	.619	.738	-	-3.19	-.103
	SOLFEM60	.616	.738	-.107	-3.17	-.104
	GALERKIN	.619	.738	-.096	-3.19	-.103
	[OT]	.62	.74	-.095	-3.24	-.103
400	FLUENT	.565	.601	-	-	-
	FIDAP	.563	.618	-	-	-
	SOLFEM30	.556	.606	-	-2.25	-.113
	SOLFEM60	.553	.606	-.123	-2.29	-.114
	GALERKIN	.556	.606	-.111	-2.29	-.114
	[OT]	.55	.60	-.112	-2.32	-.115
1000	FLUENT	.565	.557	-	-	-
	FIDAP	.531	.561	-	-	-
	SOLFEM30	.531	.563	-	-1.87	-.114
	SOLFEM60	.531	.566	-.119	-2.00	-.117
	GALERKIN	.531	.569	-.113	-2.08	-.119
	[OT]	.53	.56	-.120	-2.13	-.123
2000	FLUENT	.543	.543	-	-	-
	FIDAP	.521	.543	-	-	-
	SOLFEM30	.519	.544	-	-1.47	-.108
	SOLFEM60	.522	.547	-.110	-1.76	-.115
	GALERKIN	.525	.550	-.116	-2.02	-.122
	[OT]	.52	.54	-.143	-2.68	-.136

Table 1: Properties at vortex core

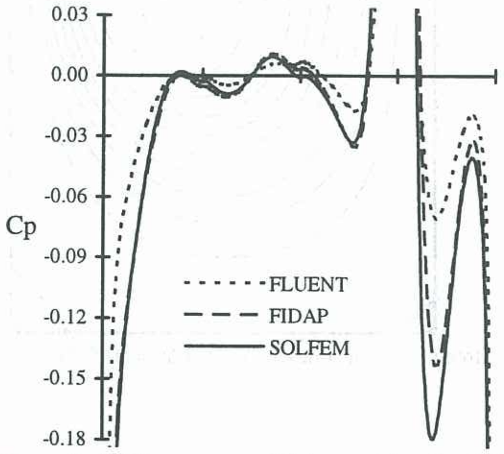


Figure 3: Wall pressure coefficient profiles, Re=100

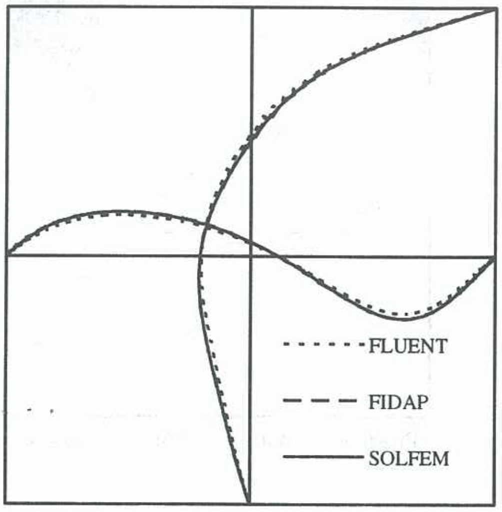


Figure 4: Centre-line velocity profiles, Re=100

CONCLUSIONS

A comparison of the SOLFEM results with those from other sources shows good agreement. At this stage, any attempt to claim more, based on these results alone, would be invalid. The purpose of this exercise was to validate the SOLFEM code, not to conduct a definitive comparison of the performance of the various codes. That is to follow. So, what then are the main features of the solenoidal approach? The solenoidal approach decouples the solution so that the velocity components are determined solely from their own values at the previous iteration. The advantage of this approach is that there is a reduction in the dimensions of the problem, with a consequent reduction in the storage requirements. This whole concept is not restricted to viscous flows, but is applicable wherever there is a similar pairing in the governing equations of a pressure-like term with a velocity-like divergence term.

REFERENCES

MACK, A.N.F., 'A unification of finite element approaches for primitive variable solutions to viscous incompressible flows', *Computational Techniques and Applications*, (Ed.) J. Noye, C. Fletcher, North-Holland, 1984

MACK, A.N.F., 'A solenoidal element for inherent mass conservation', *Computational Techniques and Applications*, (Ed.) W.L. Hogarth, B.J. Noye, Hemisphere, 1990

MACK, A.N.F., 'An element level zero-divergence finite element approach', *Int. J. for Numerical Methods in Fluids*, **19**, 795-813, 1994

NORRIE, D., DE VRIES, G., 'A survey of the finite element applications in fluid mechanics', *Finite Elements in Fluids, Vol. 3*, (Ed.) R. Gallagher, O. Zienkiewicz, J. Oden, M. Cecchi, C. Taylor, Wiley, 1978

OLSON, M., TUANN, S., 'New finite element results for the square cavity', *Computers and Fluids*, **7**, 123-135, 1979

TUANN, S., OLSON, M., 'Review of computing methods for recirculating flows', *J. of Computational Physics*, **29**, 1-19, 1978