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# EVOLUTION OF VORTICITY IN THE BATHTUB VORTEX

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## ABSTRACT

The structures of bathtub flows with weak and strong vorticities are considered. Special attention is paid to analysis of the vorticity evolution. New behavior of the stream function $\psi$ in the near-centerline region $\psi \sim r^{4/3}\Delta z$ is suggested and validated by numerical simulations.

## INTRODUCTION

Among huge variety of fluid motions there is one phenomenon - the bathtub vortex - whose existence is most obvious even for non-specialists. One needs to pull the bottom plug in a tank filled with water and, in some time, the fluid rotation becomes apparent. Inevitability of the bathtub vortex is amazing. The intensity of the vortex flow may vary significantly but it is well-known that the vortex is always present when a tank or a tub are drained through a small orifice. The basic explanation of the "magic" of the bathtub vortex is quite evident: fluid flows towards the center and the fluid rotation is amplified as it is required by the conservation of the angular momentum principle.

Although the fluid rotation in bathtub vortex can be easily explained, the bathtub vortex flow is not simple. The structure of the flow is dependent on the Rossby number whose different versions, $K$ and $E$, are introduced by

$$K=\frac{v_r^*}{(\gamma^*\omega_z^*)^{1/2}};\quad E=\frac{K}{L}=\frac{v_z^*}{(\gamma^*\omega_z^*)^{1/2}};\quad L=\frac{r^*}{z^*} \qquad (1)$$

where $v$ and $\omega$ denotes different components of velocity and vorticity, $\gamma=v_\theta r$ is circulation and "*" denotes characteristic values for the region under consideration. In the bathtub vortex flows the value of the Rossby number can be both very small and very large depending on conditions of the experiments. If at least one of the parameters $K$ or $E$ is large, the vorticity is not strong enough to change flow in $z$-$r$ plane which remains potential $\omega_\theta=0$. Such vortices shall be called "weak vortices". Alternatively, if both $K$ and $E$ are small, the vortex is strong. The bathtub vortex is, generally, time evolving and some aspects of time dependence are of prime significance for the characteristics of the flow, while other aspects can be neglected. Although the Reynolds number, Re, can be very large, the bathtub vortex flow remains laminar. The bathtub vortex is treated as inviscid by default (with certain reservations which are considered later in the paper). The bathtub flow is assumed to be axisymmetric.

The phenomenon of the bathtub vortex was repeatedly discussed in publications. Shapiro (1962) introduced explanation of the basic principles of rotational motion in the bathtub vortex. Sibulkin (1983) noted similarity of the bathtub vortex and atmospheric phenomena of much larger scales: tornadoes and hurricanes. The theory of the bathtub vortex is presented by Marris (1967) who constructed a vortex-like solution of the equations governing the time evolution of vorticity in ideal fluid assuming that $\psi=r^2\Delta z$. The solution is not bounded and should be treated as the near-centerline approximation. The solution is significant simplification of the realistic bathtub flows and is valid only under certain conditions. In addition, Marris neglected the convective terms in favor of the time derivative and this further restricts his analysis. A series of publications (Einstein & Li, 1951; Lewellen, 1962 and Lundgren, 1985) are devoted to the structure of the strong vortex flow. Among these publications, Lundgren's asymptotic analysis, which will be discussed further in the paper, is most relevant to the present consideration.

## WEAK VORTICITY IN DEEP WATER

The intensity of the circumferential component of the vorticity, $\omega_\theta$, which is present in

$$\frac{\partial^2\psi}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = -r\omega_\theta \qquad (2)$$

determines if the flow remains potential in $z$-$r$ plane. In this section we assume that rotation is relatively slow so that $\omega_\theta$ can be neglected in (2). The conditions of applicability of this assumption can be determined by analysis of the vorticity evolution. In deep water $H \gg r_b$, the flow can be divided into: a) near-sink region where $r_e \ll a \ll r_b$, $v_a=-qa^{-2}$, $\psi=q\cos(\alpha)$; b) uniform flow region $z \gg r_b$, $v_z=U=2q/r_b^2$, $\psi/q=1-(r/r_b)^2$; c) intermidiate region separating a) and b). Here, $a=r/\sin(\alpha)$ is the distance from the draining pipe whose radius $r_e$ is much smaller than the bathtub radius $r_b$, $2\pi q(t)$ is the volume flow rate and $U=-dH/dt$ where $H=H(t)$ is the level of water in the tub. The $z$-$r$ velocity can be assumed quasi-steady $\mathbf{v}_{zr}=U(t)\mathbf{V}_{zr}(r,z)$, where $\mathbf{v}_{zr}=(v_z,v_r)$.

The circulation, which remains constant for each moving fluid particle $d\gamma/dt=0$, can be determined in the near-sink region analytically when the fluid particles arrive from near-sink region or from the uniform flow region. Integration of the fluid particle trajectories yields

$$\gamma = \frac{\omega_{z0}}{2}\left(r^3 + 3\sin^3(\alpha)\tau\right)^{2/3}; \qquad \tau = \int_0^t U\,dt \qquad (3)$$

$$\gamma = \frac{\omega_{z0}}{2} r_b^2 \left(1-\cos(\alpha)\right) \qquad (4)$$

where (3) and (4) describe initial final stage of the evolution of the circulation respectively; $\omega_{z0}/2$ is the initial rotation speed.

In this section we assumed that $\omega_\theta$ is negligibly small. The generation of $\omega_\theta$ is determined by the circumferential component of the vorticity equation $d\underline{\omega}/dt = (\underline{\omega}\cdot\nabla)\mathbf{v}$ which can be written in the form (note that $(\underline{\omega}\cdot\nabla)\gamma=0$)

$$\frac{d\omega_\theta/r}{dt} = \gamma\underline{\omega}\cdot\nabla r^{-2} = -2\frac{\gamma\omega_r}{r^3} = \frac{1}{r^4}\frac{\partial\gamma^2}{\partial z} \qquad (5)$$

The evolution of the components $\omega_z$ and $\omega_r$ is controlled by equations

$$\frac{d\omega_s}{d\tau} = \frac{\omega_s}{V_s}\frac{dV_s}{d\tau} + 2\omega_n\frac{d\beta}{d\tau}; \qquad \frac{d\omega_n V_s r}{d\tau} = 0 \qquad (6)$$

where the indices "s" and "n" denotes tangential and normal components in the $z$-$r$ plane (that is $V_s=|\mathbf{V}_{zr}|$) and $\beta$ is the angle formed by the vector $\mathbf{V}_{zr}$ and the flow axis. The derivation of (6) needs some remarks. First, according to the strengthened version of the Helmholtz vortex theorem, the vortex lines are material lines and the vorticity vector $\underline{\omega}$ evolves in exactly the same way as a material vector $\delta\mathbf{x}$. That is, using conventional tensor notation, $\omega_i=(\partial x_i/\partial \mathring{x}_j)\mathring{\omega}_j$ and $\delta x_i=(\partial x_i/\partial \mathring{x}_j)\delta\mathring{x}_j$, where superscript index "o" denotes values related to the old position of the fluid particle. Hence, it is sufficient to investigate deformation and rotation of fluid elements by the velocity field and this determines evolution of the vorticity. Second, the new time $\tau$ transforms velocity $\mathbf{v}_{zr}$ into steady field $\mathbf{V}_{zr}$. The normal fluid element $\delta x_n$ is linked to the stream tube flow rate $\delta\psi = \delta x_n V_s r$ which remains constant along a streamline while the tangential fluid element $\delta x_s$ is stretched at the rate $d(\delta x_s)/d\tau=G_{ss}\delta x_s$ where $G_{ij}\equiv\partial V_i/\partial x_j$ is the velocity gradient tensor. In addition, the elongation changes angle between the fluid elements $\delta x_n$ and $\delta x_s$ so that $d(\delta x_s)=(G_{ns}+G_{sn})\delta x_n d\tau$ is projection of $\delta x_n$ on $\delta x_s$. Since $\delta x_n$ is treated as the normal component of vector $\delta\mathbf{x}$, this projection is contributed to $\delta x_s$. Considering that $\omega_\theta=G_{sn}-G_{ns}$, $d\beta=G_{ns}d\tau$ and $dV_s/d\tau=V_sG_{ss}$, we obtain

$$\frac{d(\delta x_s)}{d\tau} = \frac{\delta x_s}{V_s}\frac{dV_s}{d\tau} + 2\delta x_n\frac{d\beta}{d\tau} + \delta x_n\omega_\theta$$

Neglecting $\omega_\theta$ yields equation (6).

In the final stage of the flow, $\omega_n=0$, $\gamma=\gamma(z,r)$ and $\omega_s(\tau=0)=\omega_{z0}$. Equation (6) can be integrated: $\omega_s=V_s(\omega_s/V_s)_{\tau=0}$. If $U$=const, integration of (5) yields Batchelor's (1967) equation for steady axisymmetric motions of ideal fluid

$$r\omega_\theta = \frac{\omega_{z0}^2}{U^2}\psi - \frac{1}{2}\frac{\omega_{z0}^2}{U}r^2 \qquad (7)$$

Assuming $v_z^*\sim U$, $\omega_z^*\sim\omega_{z0}$ and $\gamma^*\sim r_b^2\omega_{z0}$, we note that if $E_b=Ur_b^{-1}\omega_{z0}^{-1}$ is large, the source term in (2) specified by (7) remains small over whole flow field including the near-sink region. Initially, $\omega_n\sim\omega_{z0}\neq0$ at the bottom of the flow and (7) is not applicable. While $\omega_n$ is rapidly decreasing in accelerating flow, $\omega_n$ may contribute to increasing $\omega_s$. According to (6) this contribution is limited by change of the flow direction, $\Delta\beta$, so that $|\omega_s/V_s|<(1+2\Delta\beta)|\omega_{z0}/V_s|_{\tau=0}$. Since $\Delta\beta$ is limited in the potential bathtub flow, $r\omega_\theta$ can not significantly exceed the value predicted by (7).

Equations (3), (4), (6) and (7) show that the deep water flow with weak vorticity does not form 2-dimensional distribution of vorticity which is typical for stronger vortices. If $E_b$ is sufficiently large, the flow in $z$-$r$ plane remains potential. In flows with smaller $E_b$, the influence of vorticity is greater. As it shown by Batchelor (1967), at $2E_b\sim3.83^{-1}$ the flow losses its ability to adjust itself to radius variations and the nature of the flow must be changed.

## THE CASE OF SHALLOW WATER

The flow in shallow water $H<<r_b$ has some specific features which are important for the formation of the bathtub vortex. As the first step, we exclude from consideration the near-centerline region $r\sim r_e$ and analyze the flow characteristics for larger $r$. It is not difficult to see that equations

$$\psi=\frac{1}{2}U\left(r_b^2-r^2\right)\frac{z}{H}, \quad v_r=\frac{1}{2}\frac{U}{H}\left(r-\frac{r_b^2}{r}\right), \quad v_z=-\frac{U}{H}z \qquad (8)$$

specify the potential ($\omega_\theta=0$) velocity which satisfy boundary conditions for inviscid flow at the bounds with exception of the centerline. The evolution of the circulation and vorticity components is given by

$$\gamma=\frac{\omega_0 r_b^2}{2}\left|1-\left(1-\frac{r^2}{r_b^2}\right)\frac{H}{H_0}\right|; \quad \omega_r=0; \quad \omega_z=\frac{H}{H_0}\omega_{z0} \qquad (9)$$

Index "0" denotes initial values and $\omega_{r0}$ is assumed to be zero. The vorticity decreases according to the third equation in (9). On the contrary, the rotational velocity $v_\theta$ increases since fluid particles move towards the centerline. On the face of the problem, this seems contradictory but, in fact, increasing of $v_\theta$ and decreasing of $\omega_z$ is an indicator of the singularity formed at the centerline. This singularity is perceived as the bathtub vortex. Near the centerline $r\ll r_b$, the flow determined by (9) is very close to two-dimensional sink flow

$$\psi \to \frac{q}{H} z \quad \text{and} \quad \gamma \to \frac{\omega_0}{2} r_b^2 \left(1 - \frac{H}{H_0}\right) \quad \text{as } r \to 0 \qquad (10)$$

The circulation near the centerline increases with time as fluid flows out. The shallow water solution is not applicable for potential flows when $H$ is large but it has much broader range of applicability when vorticity is strong. The local value of the Rossby number increases towards the centerline

$$E(r) \sim \left(\frac{r_b}{r}\right)^2 E_b \qquad (11)$$

## NEAR-CENTERLINE FLOW: WEAK VORTICITY

In this section we consider the near-centerline flow with weak vorticity in case of the shallow water $H \ll r_b$. The potential velocity field

$$\psi = b_0 + b_1 r^2 \Delta z, \; v_z = -2b_1 \Delta z, \; v_r = -b_1 r \qquad (12)$$

specifies flow in the vicinity of the point $r=0$, $\Delta z=0$ where $\Delta z = H - z$. A similar problem was considered by Marris (1967) but, unlike Marris, we do not neglect the convective terms. The evolution of vorticity is given by

$$\omega_z = \mathring{\omega}_z \exp(2b_1 t) \sim \mathring{\omega}_z \left(\frac{\mathring{r}}{r}\right)^2; \quad \omega_r = \mathring{\omega}_r \exp(-b_1 t) \sim 0 \qquad (13)$$

and the index "o" denotes initial position of the fluid particle somewhere at the border of the centerline region where $\mathring{\omega}_r$ is ~0 and $\mathring{\omega}_z$ does not depend on $z$. It should be noted that $\omega_r$ would rapidly increase if terms of higher order were not neglected in (12). The circulation can be obtained by integration of $\omega_z = r^{-1}\partial\gamma/\partial r$ giving weak logarithmic dependence on $r$ which is neglected in the following estimation of the local Rossby number considered as a function of the radius $r$

$$E(r) \sim \left(\frac{r}{\mathring{r}}\right) E(\mathring{r}) \qquad (14)$$

which obtained from $v_z$ and $\omega_z$ given by (12)&(13).

Here, unlike in the deep water flow, $\omega_z \to \infty$ as $r \to 0$ and the value of $E$ can be much smaller at the centerline $r \to 0$ than characteristic $E$ in whole near-centerline region. Thus, the strong vortex starts its formation at the centerline as rotation speed increases. The strong vortex is considered in following sections.

## STRONG VORTEX FLOW

If vorticity $\omega_z$ and circulation $\gamma$ are large ($E \ll 1$) and have the same sign $\omega_z\gamma > 0$, this has stabilizing effect on the flow. Initially, the vorticity vector is directed upwards (or downwards) and $\omega_r=0$. If some fluid particles move towards the center faster than others, than $\omega_r$ takes non-zero values so that equation (5) generates vorticity $\omega_\theta$ which forces the material elements back to vertical orientation. As it has been shown by Einstein & Li (1951), Lewellen (1962) and Lundgren (1985), the main order approximation $\Psi = \Psi_0 + \ldots$ for the stream function is given by

$$\Psi_0 = F_0(R) + \Delta Z\, F_1(R) \qquad (15)$$

The capital letters denote the dimensionless values normalized by corresponding characteristic values, that is $\Psi = \psi/\psi^*$, $R = r/r^*$, $T = t/t^*$ etc. Except for very short initial period the value of the Strouhal number $S = (r^*)^2 \omega_z^* / \gamma^*$ is small in the region $r \ll r_b$. (We emphasize that, since $\partial\gamma/\partial t = r\omega_n v_s$ in inviscid fluid, the strong vortex flow must be unsteady or viscous.) The evolution of the circulation $\gamma$ (which determines $\omega_z = r^{-1}\partial\gamma/\partial r$ and $\omega_r = -r^{-1}\partial\gamma/\partial z$) is sought in the form of the series

$$\Gamma = \Gamma_0(T) + S\Gamma_1(R,T) + \ldots + E^2 S \Gamma_2(R,Z,T) + \ldots$$

$$V_r = V_{r0}(R,T) + E^2 V_{r1}(R,Z,T) + \ldots \qquad (16)$$

The expansions of $d\gamma/dt=0$ are given by

$$\frac{\partial \Gamma_0}{\partial T} = -V_{r0}\frac{\partial \Gamma_1}{\partial R}; \quad V_{z0}\frac{\partial \Gamma_2}{\partial Z} + V_{r0}\frac{\partial \Gamma_2}{\partial R} = -V_{r1}\frac{\partial \Gamma_1}{\partial R} \qquad (17)$$

## SMALL DISTURBANCE IN THE UNIFORM FLOW WITH STRONG VORTICITY

We consider flow over small hump at $r=r_0$ whose characteristic scale $h$ is smaller than $\Delta r_K = 2^{1/2} r_0 K$ in uniform flow towards the centerline $K \ll 1$, $\psi_0 = bz$, $b$=const. The flow specified by (15) does not have enough time to preserve its structure. In the immediate vicinity of the hump $\Delta r \sim \Delta z \sim h$ the flow is controlled by the 2-dimensional Laplace equation $\nabla^2\psi=0$. In the outer zone with new independent variables $x=(r-r_0)/r_K$, $y=z/r_K$ the flow is represented by $\psi = \psi_0 + \varepsilon^2\psi_1 + \ldots$ where $\varepsilon = h/\Delta r_K$ and $\psi_1$ is governed by the equation

$$\frac{\partial^4 \psi_1}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi_1}{\partial x^4} = -\frac{\partial^2 \psi_1}{\partial y^2}; \quad k_y^2 = \frac{k_x^4}{1-k_x^2} \qquad (18)$$

which resembles the Rayleigh elastic beam equation. This equation allows solutions in form of harmonic waves whose dispersion relation is also shown in (18). Thus, a sudden disturbance in the strong vortex flow create a near-disturbance region where approximation (15) is not valid. The perturbations propagates into the rest of the flow as it is predicted by (18).

## NEAR-CENTERLINE FLOW: STRONG VORTICITY

The flow in the near-centerline region was considered by Lundgren, who assumed the approximation (15) with $F_0=0$ is valid everywhere including the flow exit. This is significant simplification of the flow. As the result, equation (15) requires that $v_r \neq 0$ at the exit. In addition, the velocity $v_z$ which determines $F_1(R)$ remains unknown and must be specified. The sink through draining pipe represents significant and sudden disturbance and, according to analysis of the previous section, should destroy (in vicinity of the exit) the flow structure specified by (15). This point of view is supported by experiments of Sakai et. al (1996) and calculations performed in the present work. In the rest of the near-centerline region (15) is applicable.

Here, we use $r^*=r_e$ and $z^*=H$ and mark corresponding $E$, $K$ and $L$ by index "e". Assuming the power asymptote at the centerline

$$\Psi_0=C_0+C_1R^{\alpha}\Delta Z,\ V_{r0}=-C_1R^{\alpha-1},\ V_{z0}=-C_1\alpha R^{\alpha-2}\Delta Z \qquad (19)$$

and using (5),(17) and $\Omega_{\theta 0}=-\partial V_{z0}/\partial R$, we determine

$$\Gamma_1=\frac{\Gamma_0'}{(2-\alpha)C_1R^{\alpha-2}};\qquad \Gamma_2=\alpha(\alpha-2)\frac{C_1^2}{\Gamma_0}R^{2\alpha-2}\Delta Z^2 \qquad (20)$$

where $\Gamma_0' \equiv d\Gamma_0/dT$. The velocity $V_{r1}$ is found from the second equation in (17) and the ratio

$$\eta \equiv \left|\frac{V_{r1}}{V_{r0}}\right| = 2\alpha(\alpha-2)\ C_1^3\frac{R^{3\alpha-4}\ \Delta Z^2}{\Gamma_0'\Gamma_0} \qquad (21)$$

is evaluated. The exit conditions direct the flow downwise and minimize the ratio $|V_r/V_z|\sim\alpha$. The values of $\alpha$ smaller than 4/3 can not be sustained by the flow since $\eta\to\infty$ as $R\to 0$. Hence, the limiting value of $\alpha$ is $\alpha=4/3$.

## AIRCORE AND VISCOUS CORE

If $\alpha=4/3$, $V_z\to\infty$ as $R\to 0$. In realistic flow, this singularity must be replaced by aircore or by viscous core (whose radius is small if Re»1). The free surface of the aircore (Lundgren, 1985; Forbes and Hocking, 1996) in a strong vortex $\Delta Z = C_2/R^2$, where $C_2 = \frac{1}{2}J^2\Gamma_0^2$ and $J^2 = v_\theta^*/(gz^*)$ is the Froude number, matches streamlines only if $\alpha=2$. Alternatively, if $\alpha\neq 2$, then $F_0\neq 0$ in (15) $\Psi_0 = C_0 + C_1R^{\alpha}(\Delta Z-C_2/R^2)$. In the viscous core (Lewellen, 1962; Lundgren, 1985), $\Gamma\to 0$ as $R\to 0$ and it is possible to show that generation of $\omega_\theta$ by (5) is not sufficient to form $V_z$-singularity at the centerline. Viscosity can also be significant factor at the bottom of the flow where viscous boundary layer is formed.

## NUMERICAL CALCULATIONS & CONCLUSIONS

Numerical simulations were performed for inviscid quasi-steady $(S\to 0)$ axisymmetric flow near the centerline $0\leq R\leq 5$, $R=r/r_e$ and involve vorticity transport calculations coupled with equation (2). The initial conditions were crucial for the stability of the time steps and overall convergence. This often required minimization of the residual by direct functional minimization which is computationally expensive for refined grid. The final convergence was reached by ~30000 small time steps.

Figure 1 shows that approximation (15), which corresponds to $V_z\sim\Delta Z$, is valid in most of the near-centerline region, but it is not valid near the flow exit. The flow is significantly disturbed here by the sink. Figure 2, plotted using the logarithmically scaled axes, demonstrates that: a) the strong vortex asymptote $\psi\sim r^{4/3}\Delta z$ is valid in relatively large region near the centerline when $E_e$ is small and b) when $E_e$ is not small, this asymptote indicates formation of the strong vortex and appears in the immediate vicinity of the centerline $R\to 0$ as it is predicted by (14).

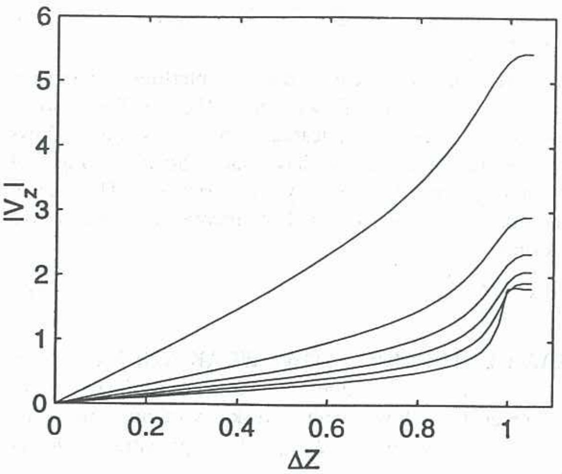


**Figure 1.** Velocity $V_z$ vs $\Delta Z$ for $K_e^2=0.1$, $E_e^2=10$, $L_e=0.1$ and $R$ = 0.04, 0.18, 0.32, 0.47, 0.63, 0.8.

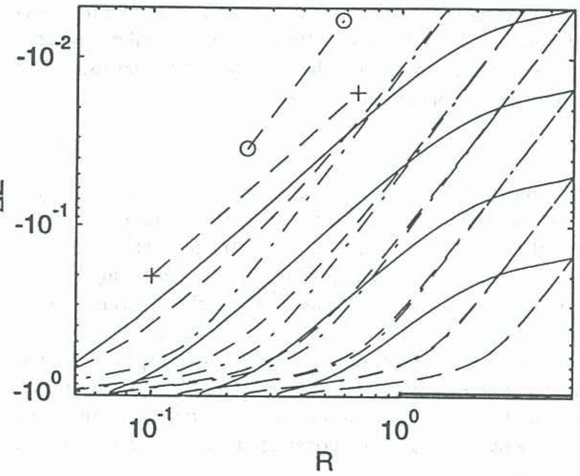


**Figure 2.** Streamlines $\Psi$=const for $L_e$=0.1 and —— $E_e^2=10$, ---- $E_e^2=5\times 10^4$, -·-·- $E_e^2=\infty$. Asymptotes o---o $\Delta Z\sim R^{-2}$ and +---+ $\Delta Z\sim R^{-4/3}$ are shown.

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