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RAPID ESTIMATION OF N-FACTORS FOR TRANSITION PREDICTION

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ABSTRACT

Accurate prediction of transition from laminar to turbulent flow is not only needed for drag estimation, but is also important in predicting scale effects when extrapolating wind tunnel data to full scale. One of the methods commonly used for estimating the most likely position of this transition is based on stability theory. Linear stability theory is used to calculate the spatial evolution of all possible instability waves. If any wave exceeds some maximum amplification the ensuing nonlinear behaviour is assumed to cause a rapid breakdown to turbulence. From a correlation of calculated amplification factors with a large amount of experimental data a value of critical amplification has been found to be roughly e^n, where n is 9. This so called "e-to-n" approach has been used for many years and although not precise it is still one of the best tools available for the purpose of estimating the location of transition. The process of calculating all the eigensolutions needed, especially in three-dimensional boundary layers, can be quite time consuming and the estimation of the transition turns out to be too expensive for routine design purposes. Here we demonstrate an approximate way of calculating eigensolutions that can be used to speed up the prediction process significantly. Approximate eigenvalues are obtained from a set of pre-computed tables covering the required range of pressure gradients, Reynolds numbers and frequencies.

THEORETICAL BACKGROUND

The stability loop defining amplification contours on a Reynolds-number~frequency-plane is a remarkably smooth function of the variables. It seems reasonable, therefore, to express the functional form of the amplification rate by a simple expansion of these two variables. Eigenvalues can then be found by evaluating the function, using previously calculated coefficients. A scheme based on a simple power series defining the growth rate in terms of two variables was first demonstrated for the case of a zero pressure gradient Blasius flow in 1977 (1). Once the coefficients of the double series had been determined accurate eigenvalues could be calculated some 1000 times as fast as by the direct solution of the Orr-Sommerfeld equation. Convergence acceleration methods had to be used to reduce the number of terms needed to provide acceptable accuracy Shanks (2). An alternative scheme, using Pade approximants (3), avoided this requirement and has also been successfully used to form tables of coefficients for a range of pressure gradients, enabling n-factor calculations to be made very rapidly. The method has been extended to three-dimensional flows (4) and is being also applied to compressible boundary layers.

The present scheme uses the same, or a related functional form, but the coefficients arising in the expressions are evaluated directly by fitting them to a set of accurate eigenvalues distributed over the domain of interest. The earlier approach defined temporal modes. It was more convenient to use a formulation that gave the spatial eigensolution directly in terms of the local Reynolds number, R_{δ^*} and frequency parameter ω involving the free-stream velocity U and the boundary layer displacement thickness δ^*. After some experimentation the form:-

$$\alpha = f(\omega, \sqrt{\tfrac{1}{R_{\delta^*}}})$$

was found to be suitable. The non-dimensional frequency parameter ω is defined as Frequency $\times\ 2\pi \frac{U}{\delta^*}$. The functional form, f, was taken as the ratio of two two-dimensional series

$$\alpha = \frac{\Sigma\Sigma A_{ij}X^iY^j}{\Sigma\Sigma B_{ij}X^iY^j},$$

where $X = Xscale \times \omega$ and $Y = Yscale \times \sqrt{\frac{1}{R}}$,

summed over any specified region i, j. The coefficients A_{ij} and B_{ij} were found by solving a set of linear equations for eigenvalues randomly distributed over the Reynolds number~frequency plane in the regions where approximate solutions were required. Coefficients were calculated for a range of Falkner-Skan velocity profiles from stagnation to separation.

The accuracy with which n-factors can be determined depends on how well the boundary layer is defined and on whether non-parallel or curvature terms are included in the formulation. Here two additional factors that can increase errors are introduced. One is the use of the Falkner-Skan family of velocity profiles. This approximation is unlikely to cause serious problems on laminar flow aerofoils, where the gradients are weak, but there maybe errors close to separation. The second approximation arises through the method of predicting the eigenvalues of the chosen family of velocity profile shapes by evaluating a function containing the previously determined coefficients. With care the errors from this cause can be reduced to a very acceptable level albeit sometimes with increased computational effort. For predicting n-factors there is no point in making unnecessarily accurate estimates of eigenvalues, because errors introduced in so doing may be far less important than those already embedded in the transition prediction scheme itself. It is important to create a sensible balance between all the sources of potential error so that reasonably useful results can be obtained with a minimum of computational effort.

CALCULATION OF TABLES

In order to illustrate the method attention is focused on the zero pressure gradient Blasius boundary layer. Figure 1 shows the spatial stability loop for eigenvalues determined by the direct solution of the Orr-Sommerfeld equation. The contours of spatial stability amplification are smooth functions of the two variables. The coefficients of the Pade type series needed to fit this function were found for various arrays sizes and layouts. After some numerical experiments it was found that a satisfactory form of array was provided by the ratio of the two functions set out above, with the arrays truncated to a triangular set of coefficients. For simplicity a simple rectangular domain was chosen for the fitting process extending to well below the critical Reynolds number and to frequencies roughly twice those of the upper branch. Figure 2 shows the stability loop constructed from the data set for the 5 X 5 pattern. Figure 3 shows a contour plot of the absolute magnitude of the differences between the approximate evaluation from the 5 X 5 array and the Orr-Sommerfeld calculation. Mostly errors within the fitting rectangle are less that 10^{-4}, but there are large regions where the errors are even smaller. Nevertheless, the accuracy was not thought to be good enough for n-factor estimation. Other arrays were tried, and generally it was found that larger array sizes reduced errors. Figure 4 shows the error contours obtained with the 9 X 9 array, where errors over most of the fitting domain have been reduced to 10^{-6}. Similar results were found for various pressure gradient parameters spanning the range from stagnation to separation. For each pressure gradient a different fitting zone was chosen to cover the most important domain for that flow. 49 pressure gradients, or values of H, were selected and the tables of coefficients evaluated. The process took some hours on a small desktop machine.

An interpolation scheme was then written, so that eigenvalues could be determined for intermediate pressure gradients. A cubic interpolation was used to enable eigenvalues for any pressure gradient, frequency and Reynolds number to be found. Generally eigenvalues were determined to an accuracy better that 1 part in 10^5 when the 9 X 9 arrays were used.

N-factor CALCULATIONS

The measured pressure distribution from an experiment carried out on an aerofoil is shown on figure 6, together with the calculated displacement thickness and shape parameter H. The velocity profiles provided by the boundary layer code were then used to make n-factor predictions by DERA (Defence Research Agency) using a code called 'CoDS'. The results of these calculations are displayed on figures 7 for a set of frequencies. The value of 'n' reaches 9 at roughly 39% chord. This position would be chosen as the probable position of the transition to turbulence. Figure 8 shows the results of the Orr-Sommerfeld calculation based on the Falkner-Skan family of velocity profiles with these values of H. There are some differences between these two predictions, but the transition position estimation is not far different from the above estimate at 41%. The calculation has also been repeated using eigenvalues calculated by the rapid estimator using the 9 X 9 arrays and the full 49 pressure tables. The results of this calculation, shown on figure 9, are again quite similar to those on the previous figures. The transition position is estimated at 40%, which is reasonably close to that of the 'CoDs' program. The calculation was repeated using a set of tables for 6 X 6 arrays. In this case only 9 pressure gradients were used to form the final data set. Figure 10 shows that these additional approximations have little effect on the final n-factor calculations.

SPEED OF EVALUATION

All the computing was carried out on an Acorn Archimedes, a 32 bit micro-computer running at a modest clock speed of 30 MHz. The speed of floating point calculations is similar to that of an Intel 486 running at 90 MHz.

In the Orr-Sommerfeld evaluation a shooting integration scheme was used and the time taken to obtain a root depended crucially on the initial eigenvalue estimate. An Orr-Sommerfeld integration using 200 steps takes 55 msecs. If it assumed that 5 iterations are needed to reach a converged result then each eigenvalue will take 0·55 secs. In the example shown on figure 6 there are 16 positions at which the eigenvalues were evaluated for 19 frequencies, making 304 eigenvalue determinations in all. The fast scheme using the full 9 X 9 array carried out this particular calculation, including the interpolations, in 62 msecs. The smaller 6 X 6 array provided the set of eigenvalues in 45. The current codes provide virtually instantaneous screen displays of growth factors and hopefully prediction of the transition position.

DISCUSSION

The proposed method of obtaining n-factors assumes that the boundary layer profiles are very close to those of the Falkner-Skan family. This is a good assumption for the region of accelerated flow up to the pressure minimum, but the match between the actual velocity profile and that from the similarity family is generally less good in highly adverse pressure gradients and is relatively poor at separation. Of course n-factors determined by any scheme is not that precise in predicting the transition process and it is therefore quite unrealistic to demand too high an accuracy for the eigenvalue prediction in the rapid scheme. The accuracy of predictions of n-factors depends on the accuracy of the series method, the interpolation scheme as well as the match to Falkner-Skan profiles. The comparisons for incompressible flows of growth curves computed directly for the true boundary are very close to those obtained by the fast scheme and suggest that the errors introduced by all these factors do not invalidate the predictions. So far this type of comparison has not been carried out on a compressible boundary layer flow, but the tools have been created to enable that to be done once proper validation of the code has been made.

The rapid scheme is so fast that it is unnecessary to consider any further reductions of computing time. In fact there is little to be gained by using the reduced array size other that in reducing the time to carry out the calculation of the coefficients.

The rapid scheme does not need any preliminary estimates of eigenvalues, making it easy to use in real applications. Also the extra overhead in calculating the growth for a very large number of frequencies is low, reducing the need to make any decisions as to the most dangerous range.

CONCLUSIONS

A scheme has been developed to calculate the coefficients of an expression that models the eigenvalue relations for any velocity profile of the Falkner-Skan family. These tables of coefficients have been used in a scheme to provide n-factors for a number of test cases. Agreement achieved with the full calculations were sufficient for transition position estimation. Comparisons with full Orr-Sommerfeld solutions for the flat plate case have shown agreement for very low Mach numbers, but progressive divergence as Mach one is approached. The differences, whether they be from coding errors or from the approximation used, may not have too large an effect on the n-factor calculations. Once the code has been validated it will be used to compute the necessary tables of coefficients for a compressible fast solver.

REFERENCES

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(3) Jiang, F. & Gaster, M. 1992 A Fast Numerical Scheme for e^n Calculations. Report to British Aerospace & Elfin.

(4) Jiang, F. & Gaster, M. 1995 'A rapid scheme for estimating transition on wings by linear stability theory', ICAS Conf. Proc., Irvine, Calif.

ACKNOWLEDGEMENTS

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Green contours are damped eigenvalues
Black contours are neutral modes
Red countours are amplified modes
Contours levels are spaced at 0.005

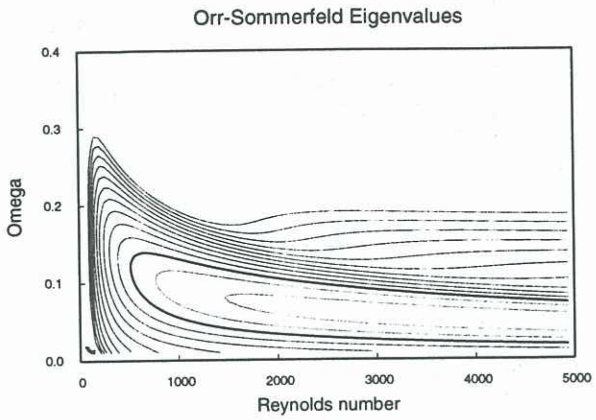

Fig 1 Spatial Amplification Contours

Error Contour Levels

1.0E-6
1.0E-5
1.0E-4
1.0E-3

The dotted blue box denotes fitting region

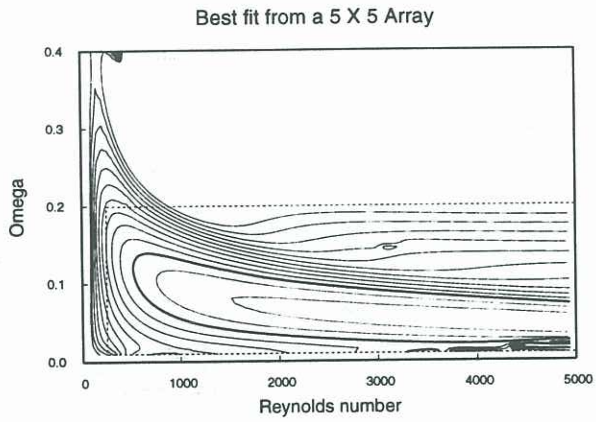

Fig 2 Spatial Amplification Contours

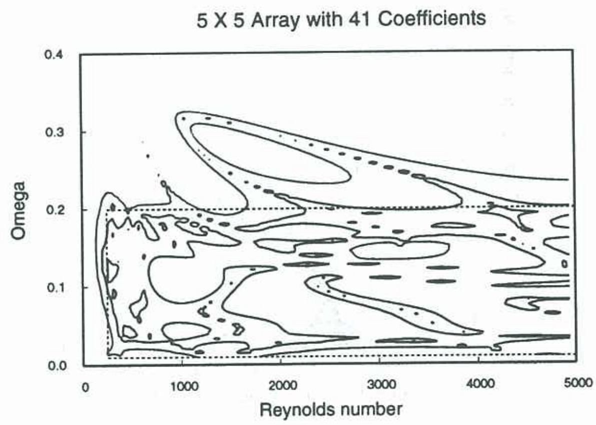

Fig 3 Amplification errors

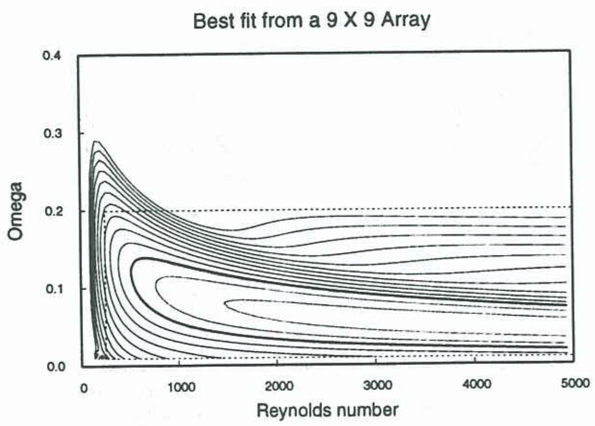

Fig 4 Spatial Amplification Contours

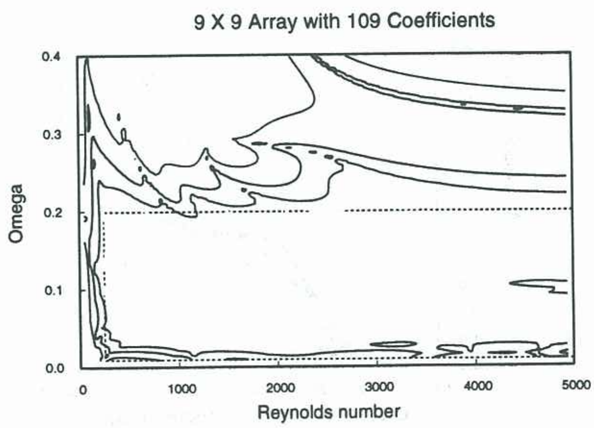

Fig 5 Amplification errors

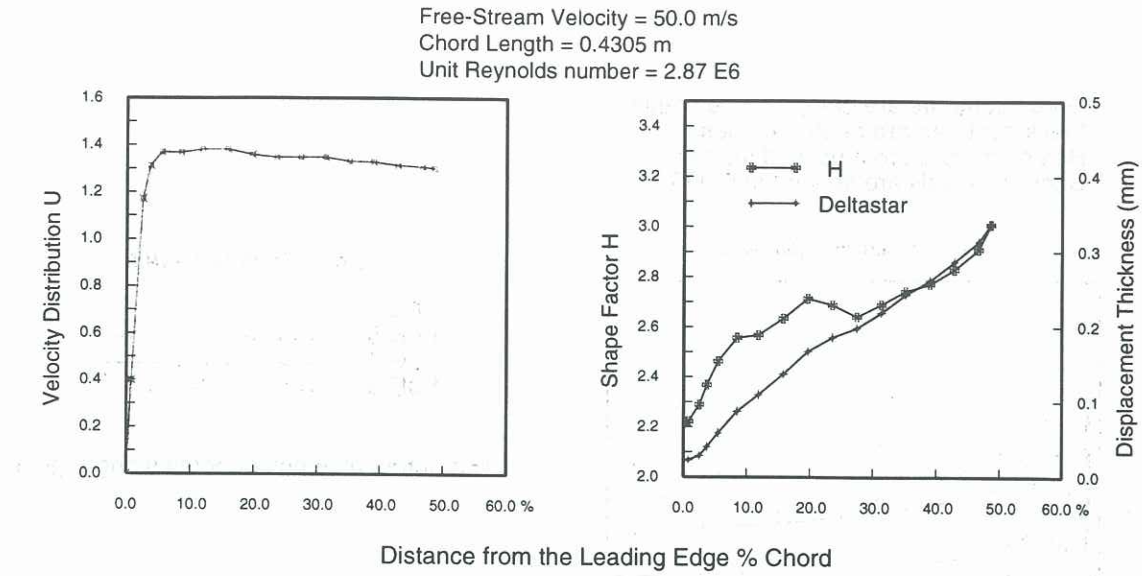

Fig 6 Boundary Layer Parameters

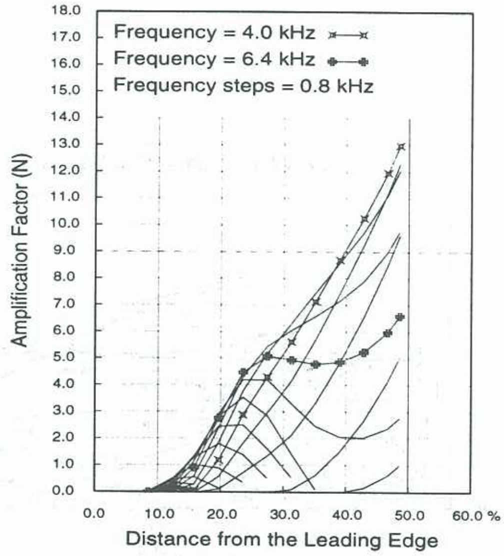

Fig 7 DERA Calculations using "CODS"

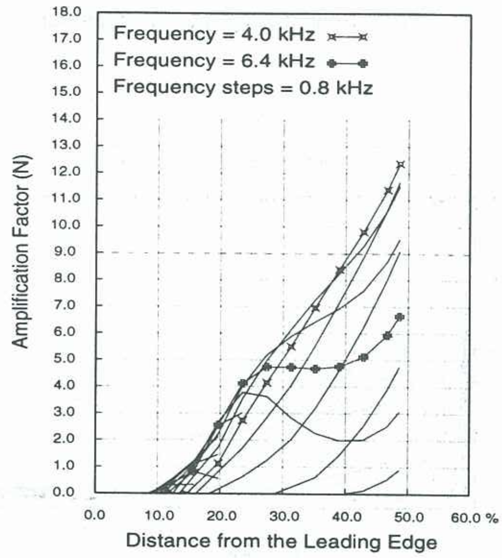

Fig 8 Falkner-Skan Eigenvalues

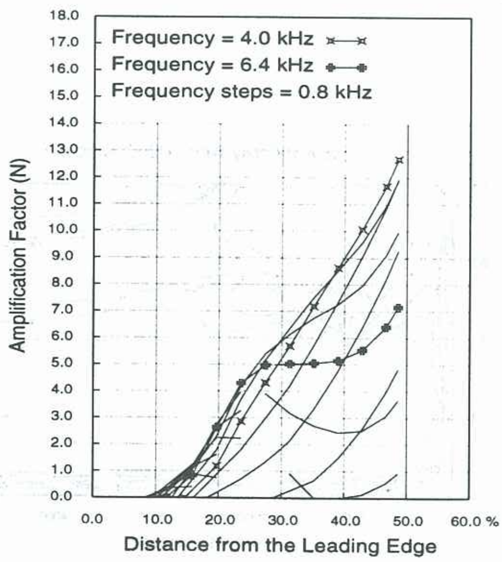

Fig 9 Rapid 9 X 9 arrays with 49 tables

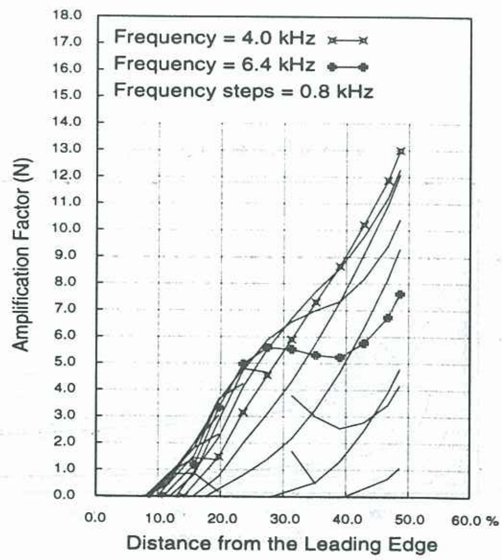

Fig 10 Rapid 6 X 6 arrays with 9 tables