13th Australasian Fluid Mechanics Conference
Monash University, Melbourne, Australia
13-18 December 1998

RESONANT OSCILLATIONS OF GAS SPHERE

Shamil U. Galiev

Department of Mechanical Engineering
The University of Auckland, Private Bag 92019, Auckland 1
NEW ZEALAND

ABSTRACT

Resonant waves may be generated in various resonators and circumstances but here, as the example, a spherical volume of gas or fluid is studied. An approximate general solution of the model equation of nonlinear acoustics is presented and used for the studying of a boundary problem. Solutions are written for resonant periodic nonlinear waves excited near and at resonance. Contributions from the nonlinear, spatial, trans-resonant, and dissipative effects are manifested by the solutions. On the one hand, expression for spherical waves is reminiscent of the solution for plane resonant waves. In particular, the Chester solution (1964) for the plane shock waves is also valid for the spherical resonant waves if dissipative effects are very small. On the other hand, because of the spatial effect, the influence of the trans-resonant and dissipative effects on nonlinear distortion of the resonant waves changes dramatically.

INTRODUCTION, GOVERNING EQUATION AND GENERAL SOLUTION

Nonlinear travelling waves were usually studied in unbounded media. There a shape of the waves is a result of a competition between the nonlinear and either dispersive or dissipative effects. The former may lead to soliton and cnoidal-type waves (solutions of the KdV-type equations). The latter may produce shock-type waves (solutions of the Burgers-type equations). In finite physical systems both left- and right-travelling waves may be excited. Near resonant frequencies the interaction of the nonlinear, dispersive, dissipative and resonant effects may be very intensive even if the excited amplitude is sufficiently small. Particularly, a balance between the nonlinear, dissipative and dispersive effects varies together with the excited frequency (Galiev and Galiev, 1998 (a)). The interaction of the left- and right-travelling waves is also important. Therefore in the trans-resonant frequency band the resonant waves may be excited which it is difficult to classify as soliton- or cnoidal- or shock-type waves.

Nonlinear resonant longitudinal waves have been studied for the last 50 years. Most of the interest has been focused on oscillations of gas in long tubes (see, for instance, Chester, 1964; Galiev *et al.*, 1970; Jimenez, 1973; Ilgamov *et al.*, 1996). It is known only a few investigations of nonlinear oscillations of gas in spherical resonators. In particular, Galiev (1971) found that the Chester solution (1964) for the plane shock waves also describes resonant spherically symmetric oscillations of some gas volumes if dissipative effects are negligible. In contrast to this conclusion Chester (1991) found that spherical shock waves did not excite in gas within resonant frequency ranges. Here we consider the Chester problem (1991) using model of thermoviscous gas. It is shown that oscillating localized spherical waves can appear in the trans-resonant ranges. These waves transform into harmonic waves far from resonant frequencies. If the dissipative effects are negligible then resonant shock waves are excited; moreover the Chester solution (1964) is valid for them.

In accordance with (Kuznetsov, 1970; Crighton, 1979)] we write an equation of nonlinear acoustics taking into account only linear and quadratic terms with respect to the velocity potential φ:

$$a^2(x^{m-1}\varphi)_{xx} - (x^{m-1}\varphi)_{tt} = x^{m-1}[a^{-2}(\gamma-1) \times\varphi_t\varphi_{tt} + 2\varphi_x\varphi_{xt} - \delta a^{-2}\varphi_{ttt}] \quad (1)$$

Here notations are standard. We will consider plane (m =1) and spherical (m =2) waves. Equation (1) is valid for media where there is a pressure-density relation. Equation, which is reminiscent of (1), is also valid for some solid media (Galiev and Galiev, 1998 (b)). An approximate solution of (1) is

$$\varphi = x^{1-m}(f_1+f_2+\psi_1+\psi_2) - \tfrac{1}{8}(2-m)a^{-1}\{(\gamma+1)[r(f_2')^2+s(f_1')^2]+2(\gamma-3)(f_1f_2'+f_2f_1')\} + \tfrac{1}{4}(m-1)a^{-1}x^{-1}\{2x^{-1}[(f_1+f_2)^2]' - \tfrac{1}{2}(\gamma+1)\times \quad (2)$$

$$\iint x^{-1}[(f_1'+f_2')^2]'\,drds\} + \tfrac{1}{4}\delta a^{-1}x^{1-m}(sf_1''+rf_2'').$$

Here $f_1 = f_1(r)$, $f_2 = f_2(s)$, $\psi_1 = \psi_1(r)$, and $\psi_2 = \psi_2(s)$, where $r = at - x$ and $s = at + x$. The prime denotes a derivative with respect to the argument. Functions f_1 and f_2 have first order; functions ψ_1 and ψ_2 have second order. The functions are unknown and must be found from initial and boundary conditions. One can consider (2) as some generalisation of the well-known d'Alembert's solution for the linear wave equation.

Let us consider surface $x = X$. Near the surface the multiplier $1/x$ under the integral in (2) is replaced by

$1/X$. As a result in (2) we have

$$\iint x^{-1}(f_1^{'}+f_2^{'})(f_1^{''}+f_2^{''})drds \qquad (3)$$

$$\approx X^{-1}[0.5s(f_1^{'})^2+0.5r(f_2^{'})^2+f_1^{'}f_2+f_2^{'}f_1]$$

For the last case solution (2) satisfies equation (1) if expression $\frac{\gamma+1}{2}a_0x^{-2}(1-xX^{-1})\,[(f_1^{'}+f_2^{'})^2]^{'}$ has third order. Thus, (3) is acceptable near surface $x = X$, where $\left|1-xX^{-1}\right| << 1$. For simplicity we assume in (2) that

$$\psi_i = \Psi_i + \tfrac{1}{4}a^{-1}r^{2-i}s^{i-1}[\tfrac{1}{2}(\gamma+1)X^{1-m}(f_i^{'})^2$$

$$-\delta f_i^{''}] + \tfrac{1}{4}(m-1)a^{-1}X^{-1}(\gamma+1)f_if_i^{'} ,$$

where $i=1$ or 2, f_1, f_2, $\Psi_1=\Psi_1(r)$ and $\Psi_2=\Psi_2(s)$ are periodic functions. In this case the secular terms are eliminated in (2). As a result, near surface $x = X$, we have for steady-state oscillations

$$\varphi = x^{1-m}(f_1+f_2+\Psi_1+\Psi_2)+\tfrac{1}{2}(m \qquad (4)$$

$$-1)a^{-1}x^{-2}[1-\tfrac{1}{4}xX^{-1}(\gamma+1)][(f_1+f_2)^2]^{'}$$

$$-\tfrac{\gamma+1}{4}a_0^{-1}x^{2-m}X^{1-m}[(f_1^{'})^2-(f_2^{'})^2]-\tfrac{\gamma-3}{4}(2-$$

$$m)a^{-1}(f_1f_2^{'}+f_2f_1^{'})+\tfrac{1}{2}\delta a^{-1}x^{2-m}(f_1^{''}-f_2^{''}).$$

BOUNDARY PROBLEM AND RESONANT SOLUTIONS

We will consider waves excited by an oscillating boundary ($x = L$). Therefore we have

$$\varphi_x = -\omega l \sin\omega t \qquad (x=L) \; ; \qquad (5)$$

$$\varphi_x = 0 \qquad (x\to 0)\, . \qquad (6)$$

For plane waves $x = 0$ in (6). For the spherical waves very close to the origin, the pressure increases and mechanical properties of the gas can strongly change. As a result equation (1) and solutions (2) and (4) are not valid if $x \approx 0$. Therefore, we consider condition (6) as a very rough approximation of reality for the spherical waves. We can rewrite this condition so that $\varphi_x = \varphi_s - \varphi_r = 0$, where (4) must be taken into account and $x \to 0$, $X \to 0$. The last equation is satisfied if

$$f_1(r)=f(r),\quad f_2(s)=(-1)^{m-1}f(s),$$

$$\Psi_1(r)=\Psi(r),\ \Psi_2(s)=(-1)^{m-1}\Psi(s) \qquad (7)$$

and $\frac{1}{r}(f_1^{'}+f_2^{'}) \to 2(f^{'})_r$ because $r\to s$. Let us consider now condition (5).

Plane Waves.

We write near the resonant frequencies that

$$f^{'}(at\pm x) = F^{'}(\xi) \pm \omega_1\omega^{-1}a^{-1}LF^{''}(\xi), \quad (8)$$

where $\xi = at \pm (x-L)$, $\omega_1 = -a^2L^{-1}\sin y$, $y=\omega La^{-1}$ and $x = L$. The boundary condition (5) becomes

$$-2\omega_1\omega^{-1}F^{''}+(\gamma+1)F^{'}F^{''}-\delta F^{'''}$$

$$= -\omega laL^{-1}\sin\omega a^{-1}(at-L+N\pi a\omega^{-1}). \quad (9)$$

Equation (9) contains the next classical acoustic solution away from resonance for some integer N:

$$f^{''} = -0.5la^{-1}\omega^2\sin\omega a^{-1}(at\pm x)/\sin y.$$

Equation (9) can be integrated once to give

$$(F^{'})^2-(\gamma+1)^{-1}[4\omega_1\omega^{-1}F^{'}+2\delta F^{''}]$$

$$= 0.5\varepsilon\cos\omega a^{-1}(at-L+N\pi a\omega^{-1})+c\,, \quad (10)$$

where $\varepsilon = 4la^2L^{-1}(\gamma+1)^{-1}$ and c is some constant of integration. One can see that equation (10) is reminiscent of equation (3.19) from (Chester, 1964). Using (8) we write the Chester solution (1964) for the travelling waves

$$f^{'}(at\pm x) = 2R\pi^{-1}\sqrt{\varepsilon}$$

$$+\sqrt{\varepsilon}\tanh[2\sqrt{q}(\sin p - R)]\cos p, \qquad (11)$$

where $p=\frac{1}{2}\omega t \pm \frac{1}{2}\omega a^{-1}(x-L)$, $R=\omega_1\pi / \omega\varepsilon^{0.5}(\gamma+1)$ and $\sqrt{q} = -\frac{1}{2}a\varepsilon^{0.5}(\gamma+1)/\omega\delta$.

Thus, according to solution (11), the weak shock waves are excited near and at resonance which travel to and fro in the gas column.

Spherical waves.

Condition (5) and (4) yield

$$L(f_2^{'}-f_1^{'}+\Psi_2^{'}-\Psi_1^{'})-f_1-f_2-\Psi_1-\Psi_2$$

$$-\tfrac{1}{4}a^{-1}L^{-1}(7-\gamma)(f_1+f_2)(f_1^{'}+f_2^{'}) \qquad (12)$$

$$+\tfrac{1}{4}a^{-1}(3-\gamma)[(f_2^{'})^2-(f_1^{'})^2-(f_1+f_2)\times$$

$$(f_1^{''}-f_2^{''})]+0.5a^{-1}L(\gamma+1)(f_1^{'}f_1^{''}+f_2^{'}f_2^{''})$$

$$-0.5\delta a^{-1}L^2(f_1^{'''}+f_2^{'''}) = -\omega lL^2\sin\omega t\,.$$

Here (7) must be taken into account. As the first approximation, it follows from (12) that

$$Lf^{'}(at-L)+Lf^{'}(at+L)+f(at-L)$$

$$-f(at+L) = \omega lL^2\sin\omega t\,. \qquad (13)$$

The solution of (13) is

$$f(at\pm x) = 0.5\omega lL^2(\sin y - \qquad (14)$$

$$y\cos y)^{-1}\cos\omega a^{-1}(at\pm x)\,.$$

From (14) we obtain resonant frequencies: $\Omega_\beta = \pi\beta aL^{-1}$, where β =1.4303; 2.4590; 3.4709; ....(Lamb, 1932). Linear solution (14) is not valid near frequencies $\omega=\Omega_\beta$. Therefore equation (12) will be considered taking into account the nonlinear terms. Following (Galiev, 1971; Galiev and Panova, 1995) we assume

$$f(at\pm x) = -L^{-1}\int F(\xi)d(\xi) \qquad (15)$$

$$-[0.5\omega_s L + (\pm)1]F(\xi),$$

where $\omega_s = 2a\omega^{-1}L^{-2}(\sin y - y\cos y)$. It is suggested for the nonlinear terms that

$$\left|aL^{-1}\int F(at)dt\right| << \left|F(at)\right|. \qquad (16)$$

We will also use the next expressions in (12):

$$\Psi(at \pm L) = -L^{-1}\int Q(at)d(at) - (\pm)Q(at)$$
$$+0.25a^{-1}(\gamma+1)[f'(at)]^2 \ . \qquad (17)$$

As the result it follows from (12)

$$\omega_s F' - \delta a^{-1}F''' - (7-\gamma)a^{-1}L^{-3}FF'$$
$$= -\omega l \sin\omega t \ . \qquad (18)$$

This equation is valid if (16) takes place. Far from a resonance solution (14) follows from (18). One can see that the basic equation for the spherical waves is different from the basic equation for the plane waves (9). Here there is the perturbed KdV-type equation instead of the perturbed Burgers-type equation (9). However if $\delta \approx 0$ we have

$$\omega_s F - \tfrac{1}{2a}(7-\gamma)L^{-3}F^2 = al\cos\omega t + c \ . \ (19)$$

For this case equations (10) and (19) practically the same. Thus, periodic spherical shock waves may be excited according to (19) and (15) if the dissipative effect is very small.

Equation (18) may be rewritten so that

$$(F - 2G\sqrt{\varepsilon_s}\pi^{-1})^2 - \tfrac{1}{4}\delta g\omega^2 a^{-3}F''_{\tau\tau}$$
$$= \varepsilon_s\cos^2\tau \ , \qquad (20)$$

where $G = \frac{1}{4}g\pi\omega_s\varepsilon^{-0.5}$, $\varepsilon_s = 2agl$, $g = 2aL^3(7-\gamma)^{-1}$, and $\tau = \omega t/2$. Following (Galiev and Galiev, 1998 (a)) we can find

$$F(p) = \sqrt{\varepsilon_s}\{2G\pi^{-1} + 3\text{sech}^2[\text{M}(\sin\text{M}^{-1}p$$
$$-G)/\sqrt{2q_0}]\cos^2 p - \cos^2 p\}, \qquad (21)$$

Here $q_0 = -0.25g\delta\omega^2 a^{-3}\varepsilon^{-0.5}$, $q_0 \ll 1$ and M =1, 2, 3,... Strictly speaking, solution (21) is valid if $G \approx 0$. Then condition (16) takes place.

According to (21) when $\sin M^{-1} p \approx G$ a peak of function $F(p)$ is generated, then a crater occurs. This excitation is reminiscent of the so-call '*oscillon*' (Umbanhowar *et al.*, 1996). By contrast to oscillons, solution (21) describes *travelling spherical oscillons.* Generally speaking, solution (21) defines a spectrum of subharmonic localised waves. If M =1 it is possible oscillations with frequency ωt.

The shape of waves (21) depends on nonlinear and dissipative properties of the medium, and on the excited frequency. The amplitude of excitation (21) is maximum at the resonance (G =0). Thus, near and at the resonance, the periodic resonant partly localised waves are predicted by (21) and (15).

Now we can write expressions for φ, pressure and velocity. However, we emphasise that (4) does not take into account the second order values far from boundaries. Therefore we must only consider first order terms in the expressions. For example we have

$$\varphi = x^{-1}[f(r) - f(s)],$$
$$\varphi_x = -x^{-1}[f'(r) + f'(s)]$$
$$-x^{-2}[f(r) - f(s)], \qquad (22)$$
$$\varphi_t = ax^{-1}[f'(r) - f'(s)] \ ,$$

where functions $f(at \pm x)$ are found, approximately, according to (21) and (15):

$$f(at \pm x) = -(\pm)\sqrt{\varepsilon_s}\{2G\pi^{-1}$$
$$+3\text{sech}^2[(\sin p - G)/\sqrt{2q_0}]\cos^2 p - \cos^2 p\}$$
$$+a\omega^{-1}L^{-1}\sqrt{\varepsilon_s}\{p - 4G\pi^{-1}p + 0.5\sin 2p$$
$$-2\sqrt{2q_0}\tanh[(\sin p - G)/\sqrt{2q_0}]\} \ . \qquad (23)$$

We put here M =1. It follows from (23) and (22) that both oscillating shock- and soliton-like waves may be excited in the sphere at the same time. However, the amplitude of the shock-like waves is small with respect to the amplitude of the soliton-like waves.

Let us consider *free* oscillations of the sphere. It follows from (18) that

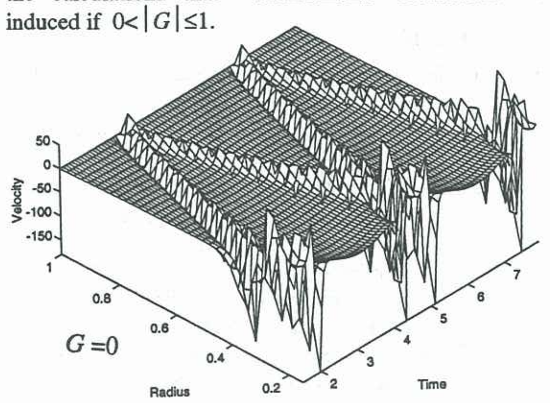
$$F'' = CF^2 - a\delta^{-1}c \ , \qquad (24)$$

where $C = -(7-\gamma)/2\delta L^3$. We assume that c is defined by an initial condition. Localised solution of (24) is

$$F(p) = (-0.66caL^3)^{0.5}[3\text{sech}^2$$
$$\times(E\sin\Omega_\beta\omega^{-1}p)\cos^2\Omega_\beta\omega^{-1}p - 1],$$
$$E^2 = -0.5Ca^2\Omega_\beta^2(-0.66caL^3)^{0.5} \ . \qquad (25)$$

Thus, if coefficient δ ensures condition $|E| \gg 1$, then localised free waves may be generated in the sphere.

In Fig. 1 results of calculations according to (23) and (22) are presented. Dimensionless time $\omega t/2$, radius (x/L) and velocity $\varphi_r\varepsilon_s^{-0.5}$ are used. We assume q_0 = 0.005 and β =1.4303. One can see that the waves of velocity have both negative and positive parts. The amplitude of the waves strongly increases very close to the origin. One might argue that this amplification in the amplitude is in practice eliminated by strongly nonlinear effects, which are not considered here. The amplitude of the waves changes, and they can form different pattens in the trans-resonant range. Oscillon-like travelling waves exist at and near the resonance (G =0 and 0.5). They form standing oscillons. If G =1 then the travelling oscillons transform into subharmonic two-peak solitons. Harmonic waves are excited if $|G|>1$. It follows from the calculations that subharmonic oscillations are induced if $0<|G|\leq 1$.



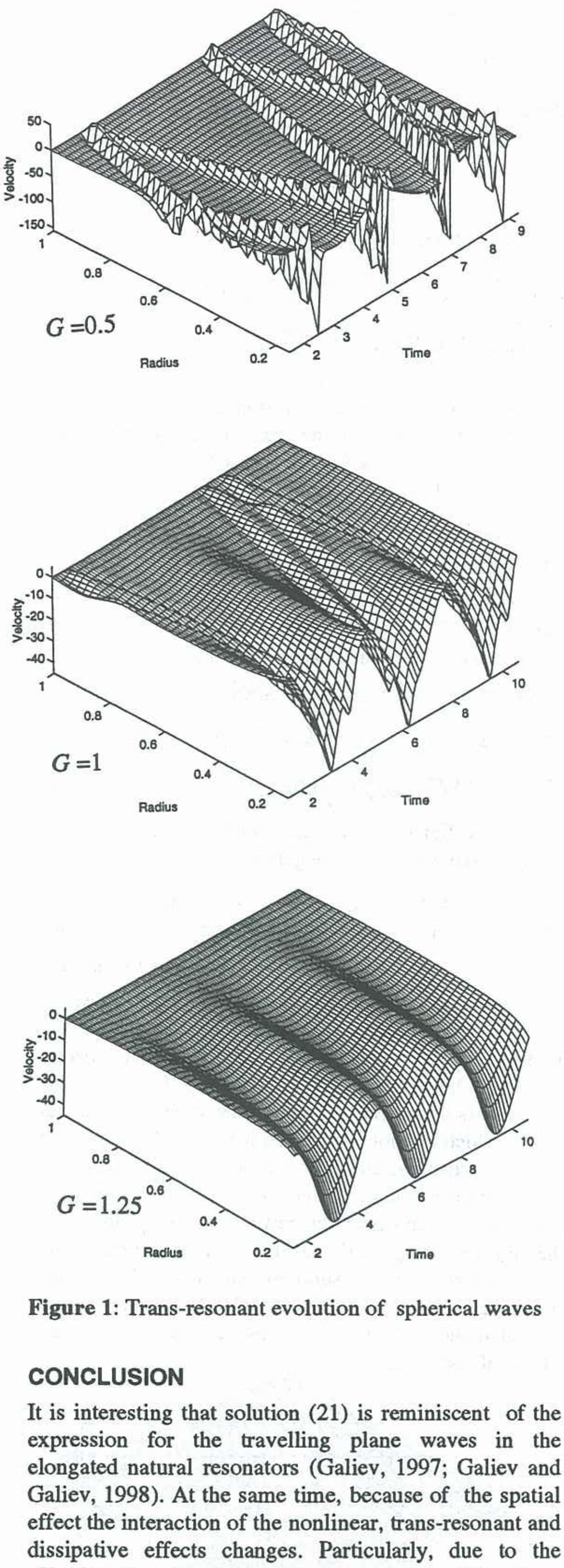


Figure 1: Trans-resonant evolution of spherical waves

CONCLUSION

It is interesting that solution (21) is reminiscent of the expression for the travelling plane waves in the elongated natural resonators (Galiev, 1997; Galiev and Galiev, 1998). At the same time, because of the spatial effect the interaction of the nonlinear, trans-resonant and dissipative effects changes. Particularly, due to the effective viscosity the term is generated in (18) whose physical role is reminiscent of the dispersive effect for the plane waves. The periodic localised oscillating waves are generated, because spatial dispersive and nonlinear effects balance each other inside the spherical waves. Localisation also takes place, because of focusing the waves. The order of amplitude $O(l^{0.5})$ of resonant spherical waves is the same as for plane waves in the elongated natural resonators (Galiev, 1997; Galiev and Galiev, 1998).

The spherical model for the simulation of different physical objects is very popular. Indeed, on the one hand, a model of a pulsating sphere is widely used in astrophysics (Gautschy and Saio, 1995). On the other hand, this model is used for studying of sonoluminescence in liquids (Wu and Roberts, 1994). We considered here some general wave properties of this model for the case of radial oscillations. The distortion of the harmonic waves into oscillating localised resonant waves was shown. At the same time our consideration was strictly limited by the aspect of the nonlinear acoustics. In particular, the presented results are not valid near the origin.

REFERENCES

CHESTER, W., "Resonant oscillations in closed tubes", *J. Fluid Mech.*, **18**, 44-64, 1964.

CHESTER, W., "Acoustic resonance in spherically symmetric waves", *Proc. R. Soc. Lond. A*, **434**, 459-463, 1991.

CRIGHTON, D. G., "Model equations of nonlinear acoustics", *Ann. Rev. Fluid Mech.*, **11**, 11-33, 1979.

GALIEV, SH. U., "Nonlinear one-dimensional oscillations of a termoviscous gas in a spherical layer", *Proc. of Seminar on Shell Theory* (Kazan Physico-Tech. Inst., Kazan, USSR), **2**, 240-253, 1971.

GALIEV, SH. U., "Resonant oscillations governed by the Boussinesq equation with damping", *Proc. 5 Int. Congress on Sound and Vibration*, Adelaide, Australia, December 15-18, 1785-1796, 1997.

GALIEV, SH. U. and PANOVA, O.P., " Periodic shock waves in spherical resonators (survey)", *Strength Mat.*, **27**, 602-620, 1995.

GALIEV, SH. U. and GALIEV, T. SH. (a), "Resonant travelling surface waves", *Phys. Lett. A*, 1998 (accepted).

GALIEV, SH. U. and GALIEV, T. SH. (b), "Spatial effect in dissipative resonant systems", *Proc. R. Soc. Lon . A* (submitted).

GALIEV, SH.U., ILGAMOV, M. A. and SADYKOV, A.V., "Periodic shock waves in a gas". *Izv. Akad. Nauk USSR, Mech. Jidcosty and Gasa* ", **2**, 57-66, 1970.

GAUTSCHY, A. and SAIO, H., "Stellar pulsations across the HR diagram: Part 1." , *Ann. Rev.Astron. Astrophys..*, **33**, 75-113, 1995.

ILGAMOV, M.A., ZARIPOV, R.G., GALIULLIN, R. G. and REPIN, V. B., "Nonlinear oscillations of a gas in a tube". *Appl. Mech. Reviews* , **49**, 137-154, 1996.

JIMENEZ, J., " Nonlinear gas oscillations in pipes. Part 1. Theory", *J. Fluid Mech.*, **59**, 23-46, 1973.

KUZNETSOV, V. P., "Equations of nonlinear acoustics", *Sov. Phys. Acoust.* , **16**, 467-470, 1970.

LAMB, H., *Hydrodynamics* (6th edn., 1932), p. 506.

UMBANHOWAR P. B., MELO, F. and SWINNEY H. L., "Localized excitations in a vertically vibrated granular layer", *Nature*, **382** , 793-796, 1996.

WU, C.C. and ROBERTS, P.H., "A model of sonoluminescence", *Proc. R. Soc. Lond. A*, **445**, 323-349, 1994.