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EVOLUTION OF VELOCITY GRADIENTS IN SIMULATIONS OF HOMOGENEOUS ISOTROPIC TURBULENCE

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ABSTRACT

Vieillefosse (1984) studied the Lagrangian evolution of the velocity gradients in terms of the restricted Euler equation where the anisotropic pressure terms and the viscous terms have been neglected. Cantwell (1992) showed that this model results in a closed system of ordinary differential equations for the second and third invariants of the velocity gradient tensor and that the evolution can be studied in terms of the solution trajectories in the phase plane of the second and third invariants. This paper shows how conditional mean trajectories can be computed using data from direct numerical simulations (DNS) of isotropic homogeneous turbulence and used to study the evolution of the invariants. These trajectories show that the contributions of the anisotropic pressure terms and the viscous terms cannot be ignored in the evolution of the velocity gradient tensor.

INTRODUCTION

The topology of the local flow pattern in a flow field is determined by the invariants of the velocity gradient tensor. Recent studies have shown that quantities related to the second and third invariants, Q_A and R_A respectively, of the velocity gradient tensor are very useful for studying structural features in simulations of turbulent flows (see Chong *al.* (1998), Blackburn *et al.* (1996), Boratav & Pelz (1995), Soria *al.* (1994) and Chen *et al.* (1990)). This paper demonstrates the possibility of studying the Lagrangian evolution of the velocity gradients in terms of quantities related to the velocity gradient tensor.

THEORETICAL BACKGROUND

A brief description of the theoretical background is described below (for a more detailed discussion, readers are referred to Chong *et al.*, 1990, Chen *et al*, 1990, and Soria *et al.*, 1994 among others). The local flow pattern as seen by a non-rotating observer moving with any point in the flow field is described by the invariants of the velocity gradient tensor $A_{ij} = \partial u_i/\partial x_j$. The invariants are solutions of the characteristic equation of A_{ij} which is given by

$$\lambda_i^3 + P_A\lambda_i^2 + Q_A\lambda_i + R_A = 0 \qquad (1)$$

where λ_i are the eigenvalues of A_{ij} and P_A, Q_A and R_A are the first, second and third invariants of A_{ij} respectively. The first invariant is given by

$$P_A = -A_{ii} \qquad (2)$$

and is zero for incompressible fluids. The second and third invariants are given by

$$Q_A = \frac{1}{2}\left(P_A^2 - A_{ij}A_{ji}\right) \qquad (3)$$

and

$$R_A = \frac{1}{3}\left(-P_A^3 + 3P_AQ_A - A_{ij}A_{jk}A_{ki}\right). \qquad (4)$$

In the $R_A - Q_A$ plane, a curve ($D = 0$) divides regions where the equation (1) has one real eigenvalue and two complex conjugate eigenvalues (i.e. when $D > 0$) from regions where equation (1) gives three distinct eigenvalues (i.e. when $D < 0$). D is the discriminant of A_{ij} and is given by

$$D = (27/4)R_A^2 + Q_A^3. \qquad (5)$$

Following the terminology of Chong *et al.* (1990), the possible four non-degenerate topologies which can be classified in the $R_A - Q_A$ plane are: stable-focus/stretching (SF/S) (when $D > 0$ and $R < 0$); unstable-focus/contracting (UF/C) (when $D > 0$ and $R > 0$); stable-node/saddle/saddle ($SN/S/S$) (when $D < 0$ and $R < 0$); and unstable-node/saddle/saddle ($UN/S/S$) (when $D < 0$ and $R > 0$).

The velocity gradient tensor A_{ij} can be split into a symmetric and a skew-symmetric component, i.e.

$$A_{ij} = S_{ij} + W_{ij} \qquad (6)$$

where $S_{ij} = 1/2(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ is the rate of strain tensor and $W_{ij} = 1/2(\partial u_i/\partial x_j - \partial u_j/\partial x_i)$ is the rate-of-rotation tensor.

The second and third invariants of S_{ij} and W_{ij} are defined in a similar fashion to the definitions of the invariants of A_{ij}.

EVOLUTION OF THE VELOCITY GRADIENTS

By differentiating the Navier-Stokes equations the following equation can be derived for the evolution of the velocity gradients $A_{ij} = \partial u_i/\partial x_j$ in a Lagrangian frame of reference, i.e. following a fluid particle in a constant density flow:

$$\frac{DA_{ij}}{Dt} = -A_{ij}A_{kj} + \frac{\delta_{ij}}{3}A_{lk}A_{kl} + H_{ij} \quad (7)$$

where D/Dt is the total derivative, δ_{ij} is the Kronecker delta function and the tensor H_{ij}, which contains the anisotropic part of the pressure Hessian and the viscous diffusion terms, is given by

$$H_{ij} = -\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x_i \partial x_j} - \frac{\partial^2 p}{\partial x_k \partial x_k}\right) + \nu\nabla^2 A_{ij}. \quad (8)$$

Vieillefosse (1984) studied the evolution of the velocity gradients by neglecting H_{ij} (this is often referred as the restricted Euler equation). Cantwell (1992) found that the restricted Euler equation led to a closed set of differential equations in the $R_A - Q_A$ plane given by

$$\frac{DR_A}{Dt} = \frac{2}{3}Q_A^2 \quad (9)$$

$$\frac{DQ_A}{Dt} = -3R_A. \quad (10)$$

In the $R_A - Q_A$ phase plane, solution trajectories of the above differential equations diverge in finite time and evolve along lines of constant discriminant D as shown in Fig. 1(a).

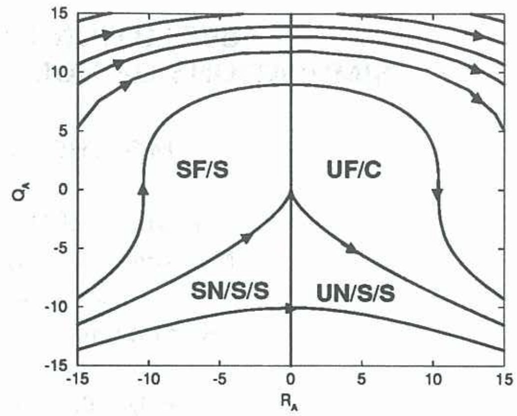
Attempts have been made to develop a more complete model by including the effects of the H_{ij} terms on the evolution of the invariants of the velocity gradient tensor. For example, Dopazo *et al.* (1993) proposed a Linear Mean Square Estimation (LMSE) model for the viscous diffusion terms in H_{ij} but ignored the anisotropic pressure terms. This model again gives a closed set of equations in the $R_A - Q_A$ plane, i.e.

$$\frac{DR_A}{Dt} = \frac{2}{3}Q_A^2 - 3\omega_0 R_A \quad (11)$$

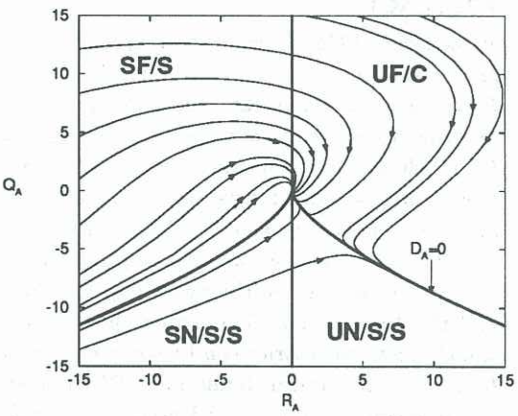
$$\frac{DQ_A}{Dt} = -3R_A - 2\omega_0 Q_A \quad (12)$$

where ω_0 is a characteristic "diffusion frequency". The trajectories corresponding to the solution of the LMSE model are shown in Fig. 1(b).

In order to investigate the possibility of developing a full model for H_{ij} one needs to study the temporal



(a)



(b)

Figure 1: Evolution of invariants. (a) Restricted Euler model. (b) LMSE model.

evolution of the invariants following a large number of random particles in the flow field. However, the tracking of a large number of particles requires enormous CPU time which is not available to the authors. An alternative method is to use some conditional averaging technique which is possible for homogeneous isotropic turbulence as the Lagrangian and Eulerian formulations are statistically equivalent for one-point statistics.

CONDITIONAL MEAN RATE OF CHANGE OF INVARIANTS

Using Eulerian data from the simulation of homogeneous isotropic turbulence, the mean vector field for the invariants (i.e. DR_A/Dt and DQ_A/Dt) is calculated by averaging the values of DR_A/Dt and DQ_A/Dt conditioned upon R_A and Q_A, i.e

$$< \frac{DR_A}{Dt} \quad | \quad R_A = R, Q_A = Q >$$

$$= \frac{1}{N} \sum_{R-\frac{\Delta_R}{2}}^{R+\frac{\Delta_R}{2}} \sum_{Q-\frac{\Delta_Q}{2}}^{Q+\frac{\Delta_Q}{2}} \frac{DR_A}{Dt} \tag{13}$$

$$< \frac{DQ_A}{Dt} \quad | \quad R_A = R, Q_A = Q >$$

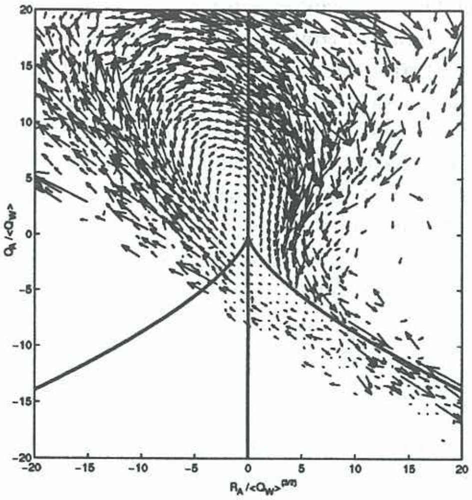
$$= \frac{1}{N} \sum_{R-\frac{\Delta_R}{2}}^{R+\frac{\Delta_R}{2}} \sum_{Q-\frac{\Delta_Q}{2}}^{Q+\frac{\Delta_Q}{2}} \frac{DQ_A}{Dt} \tag{14}$$

where Δ_R and Δ_Q are the bin size at (R, Q) and N is the number of samples in the bin. Care had to be taken that there is convergence in the statistics of DR_A/Dt and DQ_A/Dt, i.e. there must be sufficient samples in all the bins. This is dependent on bin sizes. Smaller bin sizes give higher resolution but if the bin size is too small, there is insufficient data in the bin for statistical convergence. The bin size was chosen so that there is convergence in most of the bins - roughly 300 samples are required for convergence. There were bins which had no sample and these were not used in the computations of the conditional mean trajectories.

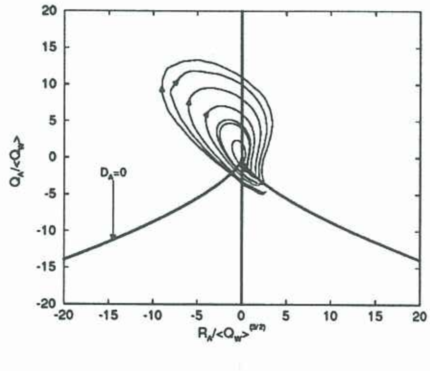
RESULTS AND DISCUSSION

The data used in this study was obtained from direct numerical simulations of forced isotropic turbulence using the spectral numerical scheme proposed by Rogallo (1981). The forcing used is similar to the forcing scheme used by Eswaran & Pope (1988) where energy is injected into the lower wave number Fourier modes as specified by an Uhlenbeck-Ornstein statistical process. The simulations were carried out for a sufficiently long time (typically several large-eddy turnover times) to ensure that the instantaneous integral characteristics are statistically steady. The data used is for the 128^3 simulation with $Re_\lambda = 70.9$. Full details of the simulations can be found in Ooi (1997).

Fig. 2(a) shows the vector field (DR_A/Dt and DQ_A/Dt), in the $R_A - Q_A$ plane. Close to the origin and on $D = 0$ curve for $R_A > 0$, the magnitude of the vectors (and hence the velocity gradients) are small. It can also be seen that the vector field in some regions of the $R_A - Q_A$ plane are not well defined due to insufficient samples. The vector field can be integrated in regions where the vector field is well defined to produce smooth conditional mean trajectories as shown in Fig. 2(b). These trajectories move in a clockwise fashion spirally towards the origin. Since decreasing R_A and Q_A imply decreasing values of velocity gradients, this suggests that fluid particles move from regions dominated by small scale motions to regions dominated by large scale motions.



(a)



(b)

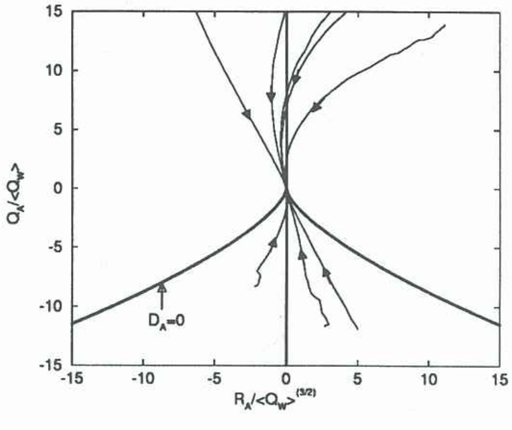
Figure 2: (a) Vector field in $R_A - Q_A$ space. (b)Integrated trajectories.

Fig. 2(b) is significantly different to Fig. 1(a) where H_{ij} has been neglected, indicating that H_{ij} has a marked effect on the Lagrangian evolution of the invariants. With the inclusion of H_{ij}, the evolution of R_A and Q_A is given by

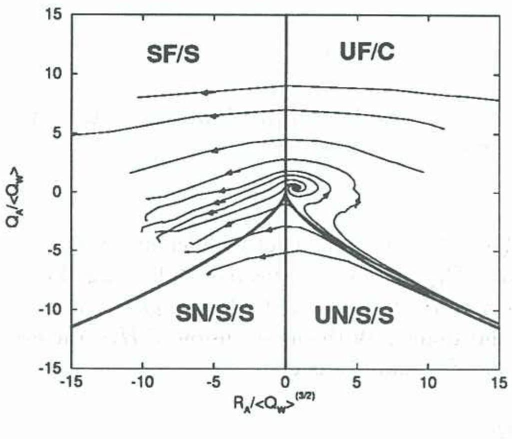
$$\begin{aligned} \frac{DR_A}{Dt} &= \frac{2}{3} Q_A^2 \\ &+ A_{in} A_{nm} \left(\frac{\partial^2 p}{\partial x_m \partial x_i} - \frac{\partial^2 p}{\partial x_k \partial x_k} \frac{\delta_{mi}}{3} \right) \\ &+ \left(-\nu A_{in} A_{nm} \frac{\partial^2 A_{mi}}{\partial x_k x_k} \right) \end{aligned}$$

$$\begin{aligned} \frac{DQ_A}{Dt} &= -3R_A \\ &+ A_{ik} \left(\frac{\partial^2 p}{\partial x_k \partial x_i} - \frac{\partial^2 p}{\partial x_m \partial x_m} \frac{\delta_{ki}}{3} \right) \\ &+ \left(-\nu A_{ik} \frac{\partial^2 A_{ki}}{\partial x_m x_m} \right) \end{aligned} \tag{15}$$

The conditional mean trajectories corresponding to the viscous terms in equation (15) are shown in Fig. 3(a). These trajectories are consistent with the LMSE model which models only the viscous terms. Fig. 3(b) shows the vector field due to the anisotropic pressure terms. It can be seen that the fluid particles experiencing a significant effect from the anisotropic part of the pressure Hessian evolve towards SF/S regions of the $R_A - Q_A$ plane. Perhaps, it may be possible to come up with a similar model for the pressure terms as for the viscous terms.



(a)



(b)

Figure 3: (a)Integrated trajectories in $R_A - Q_A$ space for viscous term. (b) Integrated trajectories in $R_A - Q_A$ space for anisotropic pressure term.

CONCLUSION

The invariants of the velocity gradient tensor not only provides an economical method of investigating the structural flow features in DNS computations of turbulent flows but is also useful in the study of how these structures evolve in a Lagrangian frame of reference. The use of conditional mean trajectories may help in the development models which will give the right dynamics for the evolution of the invariants.

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