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PRESSURE STRUCTURE FUNCTIONS IN ISOTROPIC TURBULENCE

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ABSTRACT

Attention is given to $D_p(r)$, the second-order pressure structure function in nominally isotropic turbulence. Using direct numerical simulation data for decaying isotropic turbulence, $D_p(r)$ is evaluated by a number of different methods. Calculations based on the joint Gaussian assumption are in closest agreement with the direct estimates of D_p. D_p was also calculated from longitudinal velocity fluctuation measurements in decaying grid turbulence.

INTRODUCTION

As noted by Batchelor (1951), there are many problems involving turbulent flows in which pressure fluctuations play an important role. Earlier theoretical interest focused on calculating either the two-point pressure correlations $P(r) \equiv \langle p(x)p(x+r)\rangle$ (here p is the kinematic pressure fluctuation, r is the separation between the two points and angular brackets denote averaging with respect to time), in homogeneous isotropic turbulence (Batchelor, 1951,1953; Uberoi, 1953) or the pressure structure function $D_p(r) \equiv \langle(\delta p)^2\rangle$, where the increment $\delta p \equiv p(x+r) - p(x)$ [Obukhov and Yaglom, 1951]. Solutions were possible only after assuming the Millionshchikov or joint Gaussian assumption (JGA) which allowed fourth-order velocity correlations to be replaced by second-order velocity correlations. The characteristics of pressure were also considered within the framework of the local similarity hypothesis of Kolmogorov (1941) and the assumption of local isotropy; of particular interest was the $r^{4/3}$ scaling of $D_p(r)$ in the inertial range (Obukhov, 1949; see also Monin and Yaglom, 1975, hereafter MY).

Recently, Hill and Wilczak (1995) avoided JGA and obtained an expression for $D_p(r)$, valid for locally isotropic turbulence, which contains three fourth-order velocity increment moments. Hill and Boratav (1997) and Nelkin and Chen (1998) have shown, using DNS data for forced isotropic box turbulence, that the values of $D_p(r)$, calculated using this expression, are in quite good agreement with the DNS estimates of $D_p(r)$. Hill and Boratav also indicated that JGA values of D_p were significantly smaller (by about a factor of 3) than the DNS values. This discrepancy contrasts with the reasonable agreement reported between DNS pressure spectra and JGA calculations of these spectra (e.g. Gotoh and Rogallo, 1995; Kim and Antonia, 1993). In this paper, we examine the available calculations for $D_p(r)$ (the JGA calculations have been reviewed in MY and Nelkin, 1994) against direct estimates of $D_p(r)$ obtained from a direct numerical simulation of decaying isotropic turbulence. Characteristics of the velocity field obtained from this simulation are also compared with measurements in decaying grid turbulence.

SIMULATION AND EXPERIMENTAL DETAILS

The numerical simulation was performed in a periodic box of size 2π using a finite-difference numerical method of second-order accuracy and small time steps. As a check that the energy was conserved in the inviscid limit, the energy reached the state $E(k) \sim k^2$, where $E(k)$ is the 3-D energy spectrum, in conformity with the expected equi-partition of energy, independently of the initial choice of the energy spectrum.

At $t = 0$, the spectrum

$$E(k,0) = \frac{q^2}{2A}\frac{1}{k_p^{\sigma+1}k^\sigma} exp\left[-\frac{\sigma}{2}\left(\frac{k}{k_p}\right)^2\right]$$

was prescribed, with $A = \int_o^\infty k^\sigma exp(-\sigma k^2/2)dk$, $\sigma = 2$, $k_p = 13$ and $q = 3$. For this initial input, the turbulent energy is equal to 1.138. The simulation was done by setting the Reynolds number based on the square root of this energy and on an integral scale L ($L = 1$ corresponds to the dimension 2π of the box) to 3000. This latter value was chosen empirically after carrying out a number of simulations. The final choice was that for which $k_{max}\eta$ was 1 at a dimensionless time $t = 10$. The equations were subsequently normalized by a velocity scale inferred from the initial turbulent energy and a length scale equal to $2\pi/k_p$. During the transient period, the spectrum adjusts and, at $t = 10$ ($R_\lambda \simeq 54$), it is very close to that reported by Comte-Bellot and Corrsin (1971) for $R_\lambda = 61$. At this time, the velocity derivative skewness is approximately -0.48, in good agreement with published experimental values. The simulation was performed using a 160^3 grid from $t = 10$ to $t = 50$. Over this time, the energy decayed with a power-law exponent n of 1.4 which is somewhat larger than $n = 1.2$ found by Comte-Bellot and Corrsin or $n = 1.28$ for the present experiments (see below). Results for $t = 30$ ($R_\lambda = 40$) and $t = 50$ ($R_\lambda = 36$) were saved.

X-wire measurements were made at several locations downstream of a biplane grid ($x_1/M = 20$ to 80); x_1 is measured from the plane of the grid (mesh size $M = 24.76$ mm). The grid has square rods (4.76 mm × 4.76 mm) with a solidity of 0.35. Different velocities were used ranging from about 5 m/s to ~ 22 m/s; the corresponding R_λ range was 50–100. R_λ was approximately constant for $x_1/M \gtrsim 30$ (data at $x_1/M = 50$ were used for comparison with the DNS data). Over this region, the turbulent energy decayed with a power-law exponent of 1.28.

CALCULATIONS OF $D_p(r)$

There are several ways of calculating $D_p(r)$. The simplest method is based on JGA and requires only a knowledge of either the second-order structure function $\langle(\delta u)^2\rangle$, where $\delta u \equiv u(x+r) - u(x)$ or the u-spectrum $\phi_u(k_1)$, where k_1 is the one-dimensional wavenumber. Setting $D_{LL}(r) \equiv \langle(\delta u)^2\rangle$, $D_p(r)$ can be estimated from

$$D_p(r) = \int_0^r y[D'_{LL}(y)]^2 dy + r^2 \int_r^\infty y^{-1}[D'_{LL}(y)]^2 dy \qquad (1)$$

which is deduced from the expression for $P(r)$, the two-point pressure correlation, given by Batchelor (1951,1953). Eq. (1) is given in MY, Nelkin (1994) and Hill (1994). The procedure for calculating the 1-D pressure spectrum $\phi_p(k_1)$ is outlined in Batchelor (1951,1953) and MY. $\phi_p(k_1)$ is related to the 3-D pressure spectrum $\Pi(k)$ via $\phi_p(k_1) = \int_{k_1}^\infty k^{-1}\Pi(k)dk$ and $\Pi(k)$ is given by

$$\Pi(k) = E(k)\int_0^\infty E(y) I\left(\frac{k}{y}\right) dy \qquad (2)$$

where the weighting factor I is

$$I(s) = \frac{1}{2}(s^2+s^{-2}) - \frac{1}{3} - \frac{1}{4}(s+s^{-1})(s-s^1)^2 \ln\frac{1+s}{|1-s|} .$$

$D_p(r)$ can then be inferred from $\phi_p(k_1)$ using the expression (e.g. MY)

$$D_p(r) = 2\int_0^\infty \phi_p(k_1)[1-\cos(k_1 r)]dk_1 . \qquad (3)$$

Eqs. (1) and (3) should, in principle, lead to the same result since Batchelor (1951) indicated that the assumption of joint Gaussianity of the velocity fluctuations is equivalent to Heisenberg's assumption of independence of Fourier components of velocities. The spectral calculation requires $E(k)$ to be generated. Perhaps the most accurate way of forming $E(k)$ is to start with the 1-D energy spectrum $\phi_q(k_1) \equiv \phi_u(k_1) + \phi_v(k_1) + \phi_w(k_1)$ and then apply the relation

$$E(k) = -k\left(\frac{\partial \phi_q}{\partial k_1}\right)_{k_1=k} . \qquad (4)$$

Another method of calculating $D_p(r)$ was described by Hill and Wilczak (1995); only local isotropy is assumed but a knowledge of three fourth-order velocity increments is needed, viz.

$$\begin{aligned} D_p(r) \;=\; & -\frac{1}{3}D_{LLLL}(r) + \frac{4}{3}r^2\int_r^\infty y^{-3} \\ & [D_{LLLL}(y) + D_{NNNN}(y) \\ & - 6D_{LLNN}(y)]dy + \frac{4}{3}\int_0^r y^{-1} \\ & [D_{NNNN}(y) - 3D_{LLNN}(y)]dy \qquad (5) \end{aligned}$$

the subscript N referring to a transverse direction (here y or z). Eq. (5) was simplified by Ould-Rouis et al. (1996) using empirical expressions to relate $D_{LLNN}(r) \equiv \langle(\delta u)^2(\delta v)^2\rangle$ and $D_{NNNN}(r) \equiv \langle(\delta v)^4\rangle$ to $D_{LLLL}(r) \equiv \langle(\delta u)^4\rangle$. The final result was

$$D_p(r) = \frac{r^2}{3}\int_r^\infty y^{-3} D_{LLLL}(y)dy - \frac{1}{6}D_{LLLL}(r) . \qquad (6)$$

Values of $D_p(r)$ calculated from (5) have been shown (Hill and Boratav, 1997; Nelkin and Chen, 1998) to agree with direct estimates of $D_p(r)$ from forced isotropic box turbulence DNS data. In contrast, unpublished estimates of $D_p(r)$ obtained from (5) and data (measured in several flows in our laboratory) for $\langle(\delta u)^4\rangle$, $\langle(\delta v)^4\rangle$ and $\langle(\delta u)^2(\delta v)^2\rangle$ were reasonably behaved only at small r before changing sign. A sign change also occurred for estimates obtained from (6) but at significantly larger r. The change of sign occurs when the positive contributions from $\langle(\delta u)^4\rangle$ and $\langle(\delta v)^4\rangle$ are cancelled by the negative contribution from $\langle(\delta u)^2(\delta v)^2\rangle$. This cancellation, and hence the sensitivity of Eq. (5) to the accuracy with which the 3 fourth-order moments are evaluated, was illustrated by Nelkin and Chen (1998) who rewrote (5) so as to make more explicit the contributions from $\langle(\delta u)^4\rangle$, $\langle(\delta v)^4\rangle$, $\langle(\delta u)^2(\delta v)^2\rangle$.

RESULTS

A number of tests can be applied to check isotropy over a range of scales. Isotropy at large scales should imply equality between $\langle u^2\rangle$, $\langle v^2\rangle$ and $\langle w^2\rangle$. This

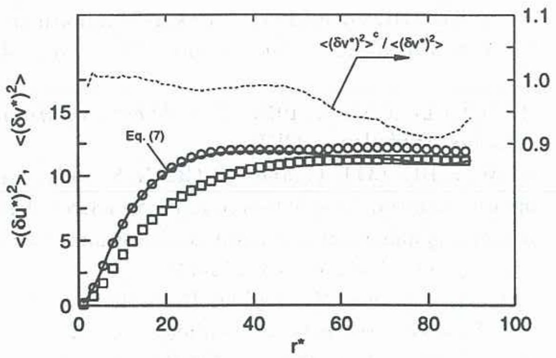

Figure 1: Distributions of $\langle(\delta u^*)^2\rangle$ (□) and $\langle(\delta v^*)^2\rangle$ (○) for the simulation at $R_\lambda = 36$. The isotropic distribution of $\langle(\delta v^*)^2\rangle^c$, calculated using (7) [the superscript c refers to the calculation] is indicated by the solid curve [a few symbols have been removed for clarity]. The ratio $\langle(\delta v^*)^2\rangle^c/\langle(\delta v^*)^2\rangle$ is also shown (- - -), on an exaggerated linear scale (right ordinate).

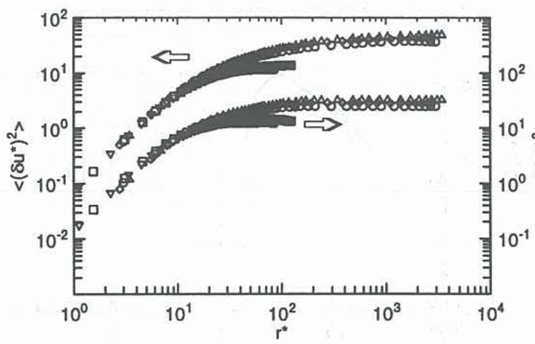

Figure 2: Measured and DNS distributions of $\langle(\delta u^*)^2\rangle$ and $\langle(\delta v^*)^2\rangle$. Note the y-origin is different for $\langle(\delta u^*)^2\rangle$ and $\langle(\delta v^*)^2\rangle$. Measurement : ○ $R_\lambda = 76$; ◇, 82; △, 101. DNS : ▽, $R_\lambda = 36$; □, 40.

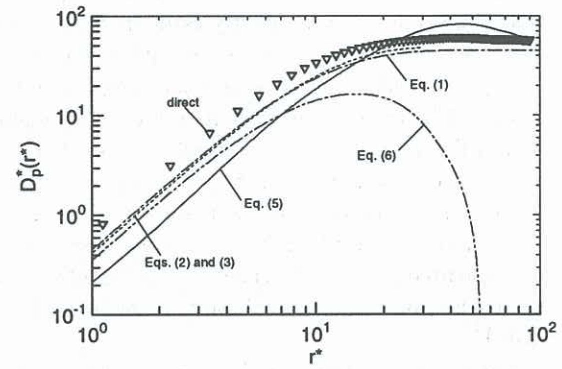

Figure 3: Comparison of calculations of $D_p^*(r^*)$ with direct estimates of $D_p^*(r^*)$. DNS data at $R_\lambda = 36$. Note that, for Eq. (5), the absolute value of $D_p^*(r^*)$ is shown.

is closely satisfied by the present simulation, e.g. $\langle v^2\rangle/\langle u^2\rangle \simeq 0.93$, $\langle w^2\rangle/\langle u^2\rangle = 1.02$ for $R_\lambda = 36$, and 0.93, 1.04 for $R_\lambda = 40$. In the experiment, the ratios $\langle v^2\rangle/\langle u^2\rangle$ and $\langle w^2\rangle/\langle u^2\rangle$ are each equal to about 0.62 (nearly independently of R_λ). A test of isotropy at any given scale r is provided by the relation between $\langle(\delta u)^2\rangle$ and $\langle(\delta v)^2\rangle$, viz.

$$\langle(\delta v)^2\rangle = \left(1 + \frac{r}{2}\frac{\partial}{\partial r}\right)\langle(\delta u)^2\rangle\,. \qquad (7)$$

Figure 1 indicates that, for $R_\lambda = 36$, this relation is satisfied for $r^* \lesssim 50$ (an asterisk denotes normalization by Kolmogorov scales); there is a departure from isotropy at larger r^* (with a maximum of about 10%). For $R_\lambda = 40$ (not shown), the relation is satisfied to better than 5% at all r. For the measured data, the relation was verified only at small r^*, although there is a tendency for the range over which agreement occurs to extend as R_λ increases.

A comparison between DNS and measured values of $\langle(\delta u^*)^2\rangle$ and $\langle(\delta v^*)^2\rangle$ is shown in Figure 2. As expected, all distributions asymptote to a constant value ($2\langle u^{*2}\rangle$ or $2\langle v^{*2}\rangle$) at large r^* and to r^{*2} at small r^*. The measured and DNS distributions appear to merge at small r^*. At large r^*, the lower locations of the DNS distributions simply reflect the lower values of R_λ for the simulations, compared to the measurements. Agreement between measurement and simulation similar to that in Figure 2 has been obtained (not shown) for fourth-order moments.

The calculations of $D_p(r)$ are compared in Figure 3 with direct estimates of $D_p(r)$ for the DNS data at $R_\lambda = 36$ (very similar results were obtained for $R_\lambda = 40$). The calculation using Eq. (1) and the DNS data for $\langle(\delta u^*)^2\rangle$, have the same shape as the directly estimated DNS distribution but with a smaller magnitude. The discrepancy is, on average, about 20% (30% for $R_\lambda = 40$). The value of $\langle p^{*2}\rangle$ inferred from the nearly constant portion of the DNS distribution at large r^*, is about 29.6 (41.1 at $R_\lambda = 40$). This corresponds to a value of the ratio $\langle p^2\rangle/\langle u^2\rangle^2$ of about 0.69 (0.74 at $R_\lambda = 40$). Previously published values of this ratio for isotropic turbulence simulations range from about 0.4 (Hunt et al., 1987) to about 1 (Schumann and Patterson, 1978; Pumir, 1994).

The calculation of D_p, based on (2), (3) and (4) is in reasonably good agreement with (1) at all values of r^*. This agreement corroborates the equivalence (Batchelor, 1951) of the assumptions made in arriving at (1) and (3). There is an almost constant difference between either (1) or (2) and the direct DNS estimates, independently of r^*. At sufficiently large r^*, JGA is reasonably well satisfied by all the data we examined; in particular, at large r, $\langle(\delta u)^4\rangle$, and $\langle(\delta v)^4\rangle$ are in good agreement with the JGA values

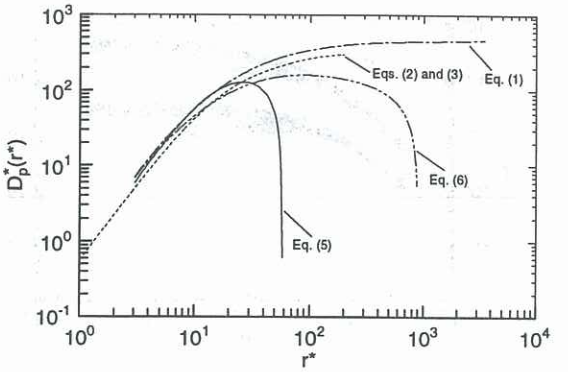

Figure 4: Calculations of $D_p^*(r^*)$ for grid turbulence measurements at $R_\lambda = 101$.

of $3\langle(\delta u)^2\rangle^2$, $3\langle(\delta v)^2\rangle^2$. Gotoh and Rogallo (1995) noted that their calculated JGA pressure spectrum was slightly below the directly estimated DNS pressure spectrum, a result consistent with Figure 3.

Eq. (5) yielded negative (and hence unacceptable) values of D_p at all r; the absolute value has been plotted in Figure 3. Eq. (6) is somewhat better behaved in the sense that it yields positive values of $D_p(r)$ across the dissipative range. It does however reach a maximum near $r^* \simeq 15$ and changes sign when the contribution from the cross term $\langle(\delta u)^2(\delta v)^2\rangle$ exceeds the sum of the contributions from $\langle(\delta u)^4\rangle$ and $\langle(\delta v)^4\rangle$.

For the grid turbulence data ($R_\lambda = 101$) in Figure 4, Eq. (5) yields positive values of $D_p(r)$ at least across the dissipative range and there is reasonable agreement over this range between Eqs. (1), (5) and (6). The significant change in behaviour for Eq. (5) between Figures 3 and 4 contrasts with the relatively good agreement between simulation and experiment of $\langle(\delta u^*)^4\rangle$, $\langle(\delta v^*)^4\rangle$ and $\langle(\delta u^*)^2(\delta v^*)^2\rangle$, i.e. the three main terms which appear in Eq. (5). The non-robustness of Eq. (5) when applied to the present data is in stark contrast with the good agreement reported by Hill and Boratav and Nelkin and Chen between Eq. (5) and direct DNS estimates of $D_p(r)$.

CONCLUSION

Of the methods considered, the calculation of $D_p(r)$ based either on the joint-Gaussian approximation or the independence of Fourier components of velocity, agrees best with the direct DNS estimates of $D_p(r)$. The calculation based on Eq. (5) is the least reliable, yielding negative values of $D_p(r)$ for all r in the case of the simulation data.

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