

## ON THE NEAR FIELD REPRESENTATION IN A FREE SURFACE VORTEX

F. Trivellato

Department of Civil and Environmental Engineering  
University of Trento  
Trento, ITALY

### ABSTRACT

The structure of the base flow and of the azimuthal flow in the near field of a free surface vortex has been herein reviewed. The knowledge of the flow field is relevant in tackling a number of problems related to the free surface vortex.

The Oseen model has been herein validated by new elaborations, covering a remarkable range of different vortex types. Besides being an exact solution of Navier-Stokes equations, the Oseen model succeeds in satisfying the vorticity balance and it appears as an exhaustive model, capable to deal with a number of features that have not been deserved much attention so far, (e.g. the evaluation of the radial length scale,  $r_m$ , of the strain rate,  $\alpha$ , and of the discharge passing through the vortex core,  $Q_v$ ). Explicit relations are herein proposed for  $r_m$ ,  $\alpha$  and  $Q_v$  and checked vs. currently available experimental data. The total discharge has been inferred to be typically as great as hundreds of times the vortex core discharge.

### THE OSEEN AZIMUTHAL MODEL.

The flow field in a free surface vortex is usually divided into two regions: the *near field*, (i.e. the viscous core region near the rotation axis) and the *far field*, (i.e. the outer irrotational domain).

No mathematical model capable of representing the structure of the flow field in a free surface core vortex has obtained so far general consensus, so that one is led to be somewhat confused by the review of the literature spanning from refined (yet unpractical or without a sounding physical basis) theories to a model as simple as the well-known Rankine vortex.

The unsteady solution describing the decaying process of a line vortex due to viscosity action was first found by Oseen in 1911; Oseen was actually the first

one to discover the exponential term of the tangential velocity, which is an essential feature to have the function behaving properly about the point  $r = 0$ . Thus eqn.(2) will be hereafter termed as *Oseen model* and it will be briefly reviewed in what follows.

Regarding the vortex flow as composed of two mutually interacting flow fields, namely the *azimuthal flow* (i.e., the tangential component of velocity,  $v$ ) and the *base flow* (at times also called secondary flow, i.e. the radial and the axial velocity components,  $u$  and  $w$ ), the Oseen approximation for the azimuthal flow comes straightly from the rigorous integration of the complete Navier Stokes equations, once an axisymmetric base flow is given in the form:

$$u(r) = -\alpha r \quad ; \quad w(z) = 2\alpha z \quad (1)$$

where  $\alpha$  is any positive constant, called *strain rate* or *stretching coefficient*. The validity of the base flow, eqn.(1), is basically limited to the near field, i.e. within a distance still to be defined at this stage (dark pointed in fig. 1).

Under the following hypothesis: incompressible, viscous fluid and steady, laminar and axisymmetric flow, the integration of the tangential component of Navier Stokes equations yields:

$$\bar{v}(\bar{r}) = \frac{1}{\bar{r}} \left( \frac{1 - e^{-1.25\bar{r}^2}}{1 - e^{-1.25}} \right) \quad (2)$$

where  $\bar{v} = v/v_m$ ,  $\bar{r} = r/r_m$ . Being a continue and derivable function, eqn.(2) definitely improves Rankine model. Following Odgaard (1986), the distance  $r_m$ , where the tangential velocity attains its maximum  $v_m$ , is given by:

$$r_m^2 \simeq 2.5 \frac{\nu}{\alpha} \quad ; \quad v_m = \frac{c}{r_m} (1 - e^{-1.25}) \quad (2.1)$$

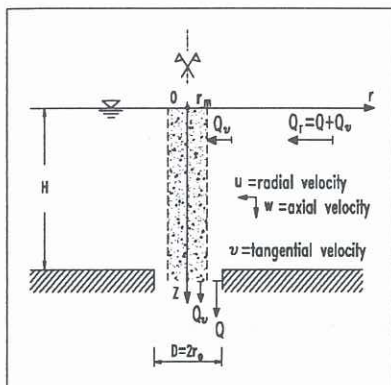


Figure 1: DEFINITION SKETCH AND NOTATION.

where  $\nu$  is the kinematic viscosity,  $c = \Gamma_\infty/2\pi$  and  $\Gamma_\infty$  is the outer circulation. Eqn.(2) has been derived according to the following boundary conditions : (a)  $r = +\infty, v = 0$  ( $rv = c$ ); (b)  $r = 0, v = 0$ .

It can be *demonstrated* the assumed base flow, eqn.(1), cannot coexist with a  $z$ -dependent azimuthal motion, and the vortex must be of infinite axial extension :  $\partial v/\partial z = 0$ .

Circulation  $\bar{\Gamma} = \Gamma/\Gamma_\infty$  is given by:

$$\bar{\Gamma} = 1 - e^{-1.25\bar{r}^2} \quad (3)$$

Vorticity is confined within a distance  $r \simeq 2r_m$  (which should actually be regarded as the proper extent of rotational flow) as it can be seen by plotting the axial component of vorticity,  $\zeta$  :  $\bar{\zeta} = e^{-1.25\bar{r}^2}$ , where  $\bar{\zeta} = \zeta/\zeta_a$ , and  $\zeta_a$  is the vorticity at the axis (i.e., at  $\bar{r} = 0$ ) :  $\zeta_a = 2.5c/r_m^2$ .

Oseen model can successfully accommodate experimental data that cover a remarkable range of different vortex types (Hite and Mih, 1994). In fig. 2 a new comparison is accomplished by elaborating Anwar's experimental data (1969), collected in a vortex of imposed circulation along the depth, satisfying therefore the condition  $\partial v/\partial z = 0$ . The data after Mory and Yurchenko (1993) are presented in fig. 2 as well ; their experimental vortex differs from Anwar's in that it is generated by suction in a rotating tank.

Also in fig. 2 a new comparison is produced by verifying the circulation vs. the experimental data by Anwar (1969). The theoretical value of circulation at  $\bar{r} = 1$  is  $\bar{\Gamma} = 0.713$ , while Anwar (1969) showed an experimental value of about 0.75. He observed that the turbulence detected in the transition region is responsible of the difference between the data and the theory based on the assumption of laminar flow.

Besides being an exact solution of Navier Stokes equations, one of the most desirable feature of Oseen model is that it does succeed in satisfying the vorticity balance, as demonstrated by De Siervi et al. (1982). And this fact should be considered more than an academic detail, as the base flow *near the vertical symmetry axis* is responsible of vorticity stretching.

Hence the base flow in the near field delivers exactly the needed flow description; to put it in another way, the base flow in the far field is not useful, as far as the determination of azimuthal flow is concerned, since it cannot stretch a vorticity that does not exist.

Having ascertained the validity of Oseen model both on theoretical and experimental basis, the model can be utilized to find relevant findings about vortex phenomenon. For example, the energy radial profile can be obtained as (Trivellato, 1995a) :

$$E = -\frac{2.5}{(1 - e^{-1.25})^2} \frac{v_m^2}{2g} S(\bar{r})$$

where  $E$  is the total head per unit weight of fluid and  $S(\bar{r})$  is an infinite series calculated by Trivellato (1995a) by applying the nonlinear transformation of Shanks (1955).

The critical submergence (headwater when the air core first reaches the intake),  $S$ , was deduced by Odgaard (1986) :  $S = 3.4(v_m^2/2g)$ ; this equation compares well with plenty of different experimental data by Odgaard (1986), Gulliver (1988), Paul (1988), Hite and Mih (1994), and it is a fine result not only in case of laminar vortex, but also it can be accurate for turbulent vortex as well: being, in fact, sufficient to replace the kinematic viscosity  $\nu$  by the effective (or total) viscosity  $\nu_e = \nu + \epsilon$ , where  $\epsilon$  is the eddy viscosity. However, the value of  $\epsilon$  is still a matter of speculation. A linear relation,  $\epsilon = k_1\Gamma_\infty$ , was first postulated by Squire (1954); Scorer (1967) proposed  $k_1 \approx 4 \cdot 10^{-4}$ ; Anwar (1969) observed eddy viscosity for turbulent vortex flow should not be constant; Odgaard (1986) estimated  $k_1 \approx 6 \cdot 10^{-5}$ , while Hite (1991) deduced  $k_1 = 0.041$ .

The approximating model by Hite and Mih (1994) can efficiently replace the Oseen model when the use of the latter becomes unpractical or even impossible : substituting eqn.(2) by an approximating function (which is amenable of exact analytical integration), it was obtained ( $z$ -axis positive downward):

$$\frac{z}{z_0} = \frac{1}{1 + 2\bar{r}^2}$$

where  $z_0$  is the air core depth. The above equation agrees well with experimental data pertaining to different types of vortex, including the one generated by the rotation of a magnetic stirring bar at the bottom floor of a closed cylinder (Julien, 1986).

## THE DISCHARGE OF THE VORTEX CORE.

Many hydraulic intakes are facing the problem of preventing floating pollution from entering the intake conduit. A number of studies have already been conducted on physical models (Heinemann and Rouve, 1977; Yildirim and Jain, 1979; Iob, 1989), so that the prediction of the flowrate directly associated to vortex motion appears to have nowadays some relevance. The base flow, eqn.(1), can efficiently address



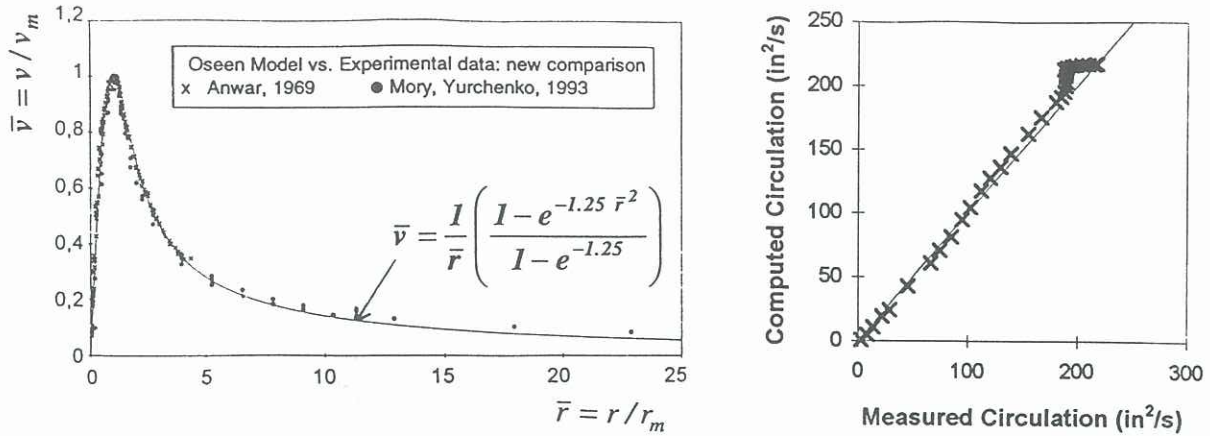


Figure 2: EQNS.(2) AND (3) VS. EXPERIMENTAL DATA : NEW COMPARISONS.

this point by adopting a method somewhat similar to the one already proposed by Odgaard (1984).

Considering a cylinder of radius  $r$  and height  $z$  (the cylinder axis being the  $z$ -axis, fig.1) the discharge,  $Q_v$ , entering the cylinder lateral surface,  $A_l$ , can be expressed as:

$$Q_v = \int_{A_l} u(r) dA_l = -2\pi\alpha z r^2$$

As a check, the discharge exiting the cylinder through the base,  $A_b$ , is given by:

$$\int_{A_b} w(z) dA_b = 2\pi\alpha z r^2$$

Choosing  $z = H$  and  $r = 2r_m$  (that is the distance where the base flow, eqn.(1), is still proper), the whole discharge passing through the vortex core is:

$$Q_v = 8\pi\alpha H r_m^2 \quad (4)$$

Combining with eqn. (2.1) :  $Q_v \simeq 20\pi\nu H$ . Alternatively :

$$R_v \simeq 20\pi \quad (5)$$

where  $R_v$  is the core Reynolds number having  $H$  as a length scale.  $Q_v$  has been measured so far in not many physical installations: in fact, not only is the core discharge tiny, but also it is hardly perceivable from the total discharge  $Q_T = Q + Q_v$ , where  $Q$  is the flowrate discharge exiting the intake hole but not through the vortex core (fig.1). The measurement of  $Q_v$  has been recently accomplished by Mory and Yurchenko (1993) in a vortex generated by suction in a rotating tank. The comparison has been performed in fig. 3 vs. the Ekman number  $E = \nu/\Omega r_o^2$  and the global Reynolds number  $R = Q_T/\nu r_o$  ( $\Omega$  is the angular velocity of the rotating cylinder;  $r_o$  is the intake radius). It should be remembered, however, that eqn.(5) pertains to an intake vortex in an infinite domain (i.e., with no imposed circulation at the border), whereas Mory and Yurchenko's vortex was

totally different in that it was generated by suction in a rotating cylinder. All considered, the comparison appears fair and suggests a reasonably large range of applicability of eqn.(5); however, more experimental evidences are clearly needed.

Taking the strain rate,  $\alpha$ , estimated according to the method shown in a follow-up paper (Trivelpato, 1995b) :

$$\alpha = \frac{MQ_T}{\pi H^3} \quad (6)$$

where  $M$  is the sum of the series :  $M = \sum_k k^{-3} \simeq 1.052$ , ( $k = 1, 3, \dots, +\infty$ ). Combining eqns. (6) and (4) :

$$\frac{Q_T}{Q_v} = \frac{1}{8M} \left( \frac{H}{r_m} \right)^2$$

It can be inferred the total discharge can typically be as great as hundreds of times the core discharge, or, in other words :  $Q_T \simeq Q$ .

#### THE RADIAL LENGTH SCALE.

An explicit relation for  $r_m$  can be derived by combining eqns.(2.1) and (6) :

$$r_m^2 \simeq \frac{2.5\pi\nu H^3}{MQ_T} \quad (7)$$

This equation is valid in an infinite flow domain, having a hole placed in the horizontal floor and with no assigned outer circulation. Unfortunately not many  $r_m$  measurements satisfying the above hypothesis are available in the literature. It appeared nevertheless of interest to have at least some rough indication about the validity of eqn.(7). Predicted  $r_m$ 's were tentatively tested vs. currently available experimental evidences in spite of the massive difference in vortex types. The first comparison is shown vs. Anwar (1969)'s data that were obtained by assigning the circulation, while computed  $r_m$ 's have no imposed circulation ( $Q_T$  in  $ft^3/s$ ;  $c = \Gamma_\infty/2\pi$  in  $in^2/s$ ) :

$r_m$ (eqn. 8)	$r_m$ (Anwar, 1969)
31 mm	47 mm ( $Q_T = 1.04$ ; $c = 312$ )
34 mm	48 mm ( $Q_T = 0.80$ ; $c = 240$ )
41 mm	49 mm ( $Q_T = 0.56$ ; $c = 172$ )
55 mm	52 mm ( $Q_T = 0.305$ ; $c = 90$ )

An agreement seems to exist at lower values of circulation, i.e. at conditions where experimental vortex does not differ too much from the theoretical one. The following comparison is performed vs. the data collected by Hite and Mih (1994), that are affected by some uncertainty due to vortex intermittency and wandering about its time-averaged rotation axis (precession motion) :

$r_m$ (eqn.8)	$r_m$ (Hite & Mih, 1994)
4.4 mm	8.8 mm
8.7 mm	7.0 mm
6.4 mm	4.9 mm

Computed  $r_m$ 's pertains to a hole in a horizontal floor and in an infinite domain, while experimental  $r_m$ 's are relative to a hole in a vertical wall, downstream of a pier.

As the last observation, neither  $\alpha$ , eqn.(6), nor  $r_m$ , eqn.(7), are suited to a mechanically driven vortex (for instance, Julien, 1986), where both the out-flow discharge and the radial velocity are clearly non-existent.

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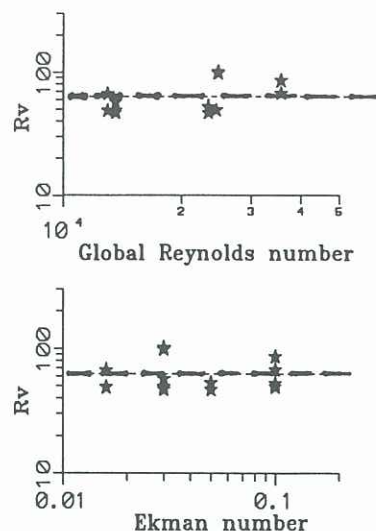


Figure 3: EQN.(5) VS. EXPERIMENTAL DATA (AFTER MORY AND YURCHENKO, 1993)

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