

A ROBUST FINITE ELEMENT AND STREAMLINE INTEGRATION METHOD FOR NON-NEWTONIAN FLUID FLOWS: AVSS/SI

Junsuo Sun, Nhan Phan-Thien and Roger I. Tanner
Mechanical & Mechatronic Engineering Department
The University of Sydney
Sydney, New South Wales
Australia

ABSTRACT

We reported an adaptive viscoelastic stress splitting and streamline integration method (AVSS/SI), and applied the method to compute the flow past a sphere in a tube filled with a Maxwell fluid. Convergent solutions were obtained up to a Weissenberg number of $O(2.8)$, representing a significant improvement over the previous streamline integration method, which ceased to converge at a Weissenberg number of $O(0.3)$. The results compared favourably with those obtained by different techniques.

INTRODUCTION

Significant developments of viscoelastic computational methods have been made in the last decade (Crochet, 1989; Brown and McKinley, 1994). Four relatively accurate and stable methods have been noteworthy in the literature: the Explicitly Elliptic Momentum Equation formulation (EEME) (King *et al.*, 1988), the Elastic Viscous Split Stress formulation (EVSS) (Rajagopalan *et al.*, 1990), the consistent Streamline Upwind/Petrov-Galerkin method (SUPG 4×4) (Marchal and Crochet, 1987), and the high-order finite element method (Talwar and Khomami, 1992). With these methods, convergent solutions for the flow past a sphere in a tube filled with a Maxwell fluid have been demonstrated up to a Weissenberg number of at least 1.5 ($W_i \leq 1.6$ by Lunsman *et al.*, 1993; $W_i \leq 2.2$ by Jin *et al.*, 1991; $W_i \leq 1.5$ by Crochet and Legat, 1992; $W_i \leq 1.6$ by Khomami, 1993). Recently, Fan and Crochet (1995) developed an implementation of the EVSS using higher-order finite ele-

ments and a modified version of the streamline upwind Petrov-Galerkin (SUPG) method. The flow past a sphere in a tube was demonstrated to be p -convergent; and the convergence region was extended up to $W_i = 2.1$. More recently, with the EVSS formulation, Luo (1995) obtained a convergent solution of the same problem up to $W_i = 2.8$ using a transient algorithm, in which the operator splitting and SUPG (OS/SUPG) methods are used for the kinematics and stress calculations respectively.

In solving the constitutive equation for a viscoelastic fluid (which is hyperbolic in character), other methods besides SUPG may also be suitable, such as streamline integration methods. In streamline integration schemes dealing with integral constitutive equations, the stress is integrated along the streamline of a fluid particle. In the early work of Viriyayuthakorn and Caswell (1980), the deformation history was computed on the basis of the Lagrangian deformation of each element of the mesh. The difficulty of this method is that badly distorted elements may lead to an inaccurate stress calculation. This has been largely overcome in later work, for example, the Streamline Finite Element Method (SFEM) developed by Luo and Tanner (1986), and the method developed by Dupont and Crochet (1985). In the SFEM (Luo and Tanner, 1986) the deformation history is integrated along the existing streamlines connecting element nodes where the viscoelastic stresses are to be evaluated. Indeed, the SFEM maintains accuracy and efficiency in the stress integration; however, it is very inconvenient in problems with

recirculating regions. This drawback was circumvented by Luo and Mitsoulis (1990) using conventional quadratic elements in the finite element calculation. Sun and Tanner (1994) further improved the SFEM, using conventional triangular elements so that an unstructured mesh can be adopted, which allows complex flow geometries to be easily simulated. In the evaluation of the deformation history, both particle tracking and Finger strain tensor calculations are performed by using a fourth-order Runge–Kutta method. In the integration of non-Newtonian extra stresses, three Gauss–Laguerre quadratures (16 points, 32 points and 68 points) are used so that thin stress layers often encountered in complex viscoelastic flow can be captured. Although streamline integration algorithms are, in principle, quite suitable in solving the constitutive equation of hyperbolic type, the highest viscoelastic level reached in the benchmark flow past a sphere in a tube filled with a Maxwell fluid is only $O(0.3)$ (see, e.g., Sun and Tanner, 1994).

In this paper, we focus on how to split adaptively the viscoelastic stress to obtain convergent solutions at high Weissenberg numbers. Then, we will use an adaptive viscoelastic stress splitting and streamline integration method (AVSS/SI) to calculate the benchmark flow past a sphere in a tube.

MATHEMATICAL FORMULATION

Governing Equations

For a steady state, isothermal, incompressible and creeping flow of a Maxwell fluid, the governing equations are the momentum and continuity equations,

$$\nabla \cdot \boldsymbol{\tau} - \nabla P = 0, \quad (1)$$

$$\nabla \cdot \mathbf{V} = 0, \quad (2)$$

and the constitutive equation, written in integral form as

$$\boldsymbol{\tau} = \int_{-\infty}^t \frac{\eta}{\lambda^2} \exp\left(-\frac{t-t'}{\lambda}\right) \left(\mathbf{C}^{-1}(t-t') - \boldsymbol{\delta}\right) dt', \quad (3)$$

where \mathbf{V} , P and $\boldsymbol{\tau}$ are the velocity vector, pressure and extra stress tensor, respectively; λ and η are the relaxation time and zero shear rate viscosity of the constitutive fluid, respectively; \mathbf{C}^{-1} is the right relative Finger strain tensor; and $\boldsymbol{\delta}$ is a unit tensor. Eqs. (1)–(3), together with appropriate boundary conditions, complete the mathematical descriptions of the problem to be solved. For most viscoelastic flows of practical interests and theoretical importance, these sets of governing equations are quite difficult, if

not impossible, to solve analytically without significant simplification. Hence, numerical methods are required.

Numerical Method

There are two widely used methods in the treatment of the momentum equation for viscoelastic flows. One is the EEME (King *et al.*, 1988). However, it is very difficult to implement the EEME formulation with an arbitrary constitutive equation; moreover, the traction boundary condition cannot be simply applied in the discretised EEME formulation. The other method is the EVSS (Rajagopalan *et al.*, 1990), in which the viscoelastic stress is split into

$$\boldsymbol{\tau} = \boldsymbol{\tau}^e + \boldsymbol{\tau}^v, \quad (4)$$

where $\boldsymbol{\tau}^e$ denotes the elastic part of the viscoelastic stress and

$$\boldsymbol{\tau}^v = 2\eta_a \mathbf{D} \quad (5)$$

represents the viscous part, where $\eta_a = \eta$ and $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{V} + \nabla \mathbf{V}^T)$ is the rate of deformation tensor. By substituting Eqs. (4) and (5) into the governing equations, Eqs. (1)–(3), we arrive at

$$2\nabla \cdot (\eta_a \mathbf{D}) - \nabla P + \nabla \cdot \boldsymbol{\tau}^e = 0, \quad (6)$$

$$\nabla \cdot \mathbf{V} = 0, \quad (7)$$

and

$$\begin{aligned} \boldsymbol{\tau}^e = & \int_{-\infty}^t \frac{\eta}{\lambda^2} \exp\left(-\frac{t-t'}{\lambda}\right) \\ & \left(\mathbf{C}^{-1}(t-t') - \boldsymbol{\delta}\right) dt' - 2\eta_a \mathbf{D}. \end{aligned} \quad (8)$$

Although converged solutions up to a Weissenberg number of $O(1.6)$ or higher have been obtained with the EVSS scheme, using by a coupled method, the highest viscoelastic level reached with the EVSS and a streamline integration method, in which a Picard iterative scheme is used, is still a low Weissenberg number of $O(0.3)$. This lack of convergence is due partly to the decoupled nature of the solution method, and partly to the high level of elastic stress. To illustrate this, we consider an element K_e as an example. We normalize the coordinates, velocity, pressure and stress according to

$$\bar{\mathbf{x}} = \frac{\mathbf{x}}{h_e}, \quad \bar{\mathbf{V}} = \frac{\mathbf{V}}{|\mathbf{V}|_{\max}}, \quad \bar{P} = \frac{P}{|P|_{\max}},$$

$$\bar{\boldsymbol{\tau}}^e = \frac{\boldsymbol{\tau}^e}{|\tau_{ij}^e|_{\max}}, \quad \text{in } K_e,$$

where \mathbf{x} represents the coordinates, h_e is the characteristic size of the finite element K_e , $|\mathbf{V}| = (\mathbf{V} \cdot \mathbf{V})^{\frac{1}{2}}$, and $|\cdot|_{\max}$ represents the maximum

magnitude of the variable concerned. Hence, the momentum equation, Eq. (6), can be written as

$$\frac{\epsilon_e}{h_e} \nabla^2 \bar{\mathbf{V}} - \frac{|P|_{\max}}{|\tau_{ij}^e|_{\max}} \nabla \bar{P} + \nabla \cdot \bar{\boldsymbol{\tau}}^e = 0, \quad (9)$$

where

$$\epsilon_e = \frac{|\mathbf{V}|_{\max}}{|\tau_{ij}^e|_{\max}} \eta_a, \text{ in } K_e. \quad (10)$$

From Eq. (9) it can be easily seen that if $\frac{\epsilon_e}{h_e} \gg 1$, the momentum equation is dominated by the viscous terms $\frac{\epsilon_e}{h_e} \nabla^2 \bar{\mathbf{V}}$ and we expect the kinematics calculation to be stable. If $\frac{\epsilon_e}{h_e} \sim O(1)$, the viscous and elastic ($\nabla \cdot \bar{\boldsymbol{\tau}}^e$) terms are of the same order; and if $\frac{\epsilon_e}{h_e} \ll 1$, i.e. $\epsilon_e \ll h_e$, the momentum equation is dominated by the elastic terms and the kinematics calculation is expected to be overly sensitive to small changes in the elastic stress.

To alleviate the difficulty caused by the sensitivity of the kinematics to the stress calculations, viscoelastic stress splitting must be adaptive, i.e., the value of viscous stress $\boldsymbol{\tau}^v$ must have, at least, the same order as that of the elastic stress $\boldsymbol{\tau}^e$,

$$|\tau_{ij}^v|_{\max} \geq |\tau_{ij}^e|_{\max}, \text{ in } K_e, \quad (11)$$

By replacing $|\tau_{ij}^v|_{\max}$ with $\eta_a \frac{|\mathbf{V}|_{\max}}{h_e}$ we arrive at

$$\eta_a = \alpha \frac{h_e |\tau_{ij}^e|_{\max}}{|\mathbf{V}|_{\max}}, \text{ in } K_e, \quad (12)$$

where $\alpha \geq 1$ is a parameter. Now, let us re-examine the momentum equation, Eq. (9), by substituting Eq. (12) into Eq. (10); it is found that $\epsilon_e = \alpha h_e \geq h_e$; hence, if this equation is solved iteratively in a decoupled manner, the sensitivity of the kinematics to small changes in the elastic stress may be avoided. We call this scheme, which splits the viscoelastic stress into viscous and elastic components, in which the viscous part assumes an adaptive value depending on the magnitude of the elastic part, the *adaptive viscoelastic stress splitting* (AVSS). If η_a is kept constant at η , the EVSS formulation is recovered.

The governing equations with the application of the AVSS, Eqs. (6)–(8), can be solved by using the mixed finite element and streamline integration method (for the detailed implementation, the reader is referred to Sun and Tanner (1994)). We call this method the AVSS/SI.

FLOW PAST A SPHERE IN A TUBE

We test the AVSS/SI by solving the benchmark flow past a sphere in a tube filled with a Maxwell

fluid. The ratio of tube to sphere radius is 2, and the geometry and the boundary conditions have been well defined in the literature (see, e.g., Lunsmann *et al.*, 1993). One of the interesting quantities in flows past a sphere is the drag coefficient, defined as $D_c = \frac{F}{6\pi}$ where F is the drag force on the sphere. A comparison of the predicted values of D_c by the AVSS/SI using the finest mesh that we could afford and those recently found in the literature is shown in Fig. 1. When $W_i \leq 1.5$, our results agree well with those found in the literature. The results for $1.5 \leq W_i \leq 2.2$ are in between the drag coefficient values reported by Jin *et al.* (1991) and those by Luo (1995). For $2.2 \leq W_i \leq 2.8$, our results are higher than Luo's results. By extrapolating D_c to the polar mesh size of $0.5h_3$, where h_3 is the polar mesh size of finest mesh, we found the drag coefficient decreases monotonically with increasing W_i , and tends to reach an asymptote which falls in the region of $4.00 \leq D_c \leq 4.03$, close to the asymptotic value of about 4.02 reported by Fan and Crochet (1995). The viscoelastic stresses predicted at $W_i = 2.0$ are tabulated in Table 1. Our results are in good agreement with those by Fan and Crochet (1995), but differ to Luo's results (1995).

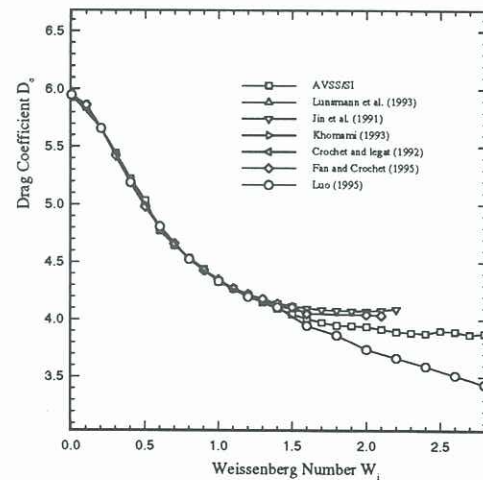


Figure 1: The predicted drag coefficient versus the Weissenberg number W_i .

CONCLUSIONS

The standard viscoelastic stress splitting scheme, when used in a decoupled manner, can cause an oversensitivity in the kinematics with small changes in the elastic stress, which leads to numerical divergence. This can be largely alleviated by the proposed AVSS scheme. The pre-

	$\frac{\tau_{zz,min}}{\tau_{zz,max}}$	$\frac{\tau_{rz,min}}{\tau_{rz,max}}$	$\frac{\tau_{rr,min}}{\tau_{rr,max}}$
AVSS/SI	$\frac{-0.425}{134}$	$\frac{-44.4}{73.3}$	$\frac{-0.366}{73.4}$
Fan & Crochet	$\frac{-0.427}{125}$	$\frac{-45.2}{77.2}$	$\frac{-0.363}{80.2}$
Luo	$\frac{-0.552}{97.1}$	$\frac{-26.0}{35.7}$	$\frac{-0.572}{32.9}$

Table 1: The maximum and minimum values of the extra stress predicted at $W_i = 2.0$ by different methods.

vious limitation of $W_i \leq O(0.3)$ in the streamline integration scheme was caused mainly by this oversensitivity (Sun and Tanner, 1994). The numerical results show that the streamline integration method is actually more stable than the SUPG, and the proposed AVSS/SI, which extended the convergence region for the solution of the flow past a sphere problem from $W_i \leq O(0.3)$ to $W_i \leq O(2.8)$, is robust.

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