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ANALYTIC SOLUTIONS FOR SOLUTE TRANSPORT IN HILLSIDE SEEPAGE

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ABSTRACT

The advection and dispersion of solute in groundwater systems can have a severe environmental impact. In this paper, we consider steady, neutrally buoyant solute transport through the saturated region of a homogeneous hillslope aquifer. Analytic series methods are used to obtain solutions for the potential and stream functions, and hence the flow field, on arbitrary seepage domains. The potential and stream functions provide a natural orthogonal curvilinear coordinate system for the solution of the transport equation. In this coordinate system, the transport equation for steady, isotropic diffusion reduces to a Helmholtz equation, with constant coefficients. An analytic series solution is obtained for the transport equation, in the transformed domain. These solutions are then transformed back to the original seepage domain, and solutions are provided for typical hillside seepage regions.

INTRODUCTION

In an environmentally conscious society, the management and conservation of subsurface water resources is extremely important. Effective management policies depend on quantitative knowledge of the transport processes in porous media. In particular, knowledge of the advection and diffusion of solutes through saturated aquifers is of prime importance in a relatively dry country like Australia, where groundwater is an essential natural resource. Solute transport can occur when the water table rises and mobilises salts in the newly saturated zone. For example, increases in the water table elevation can be caused by the removal of large surface vegetation, followed by the introduction of irrigated agriculture. More generally, contaminants can be carried from any surface source, through the vadose zone and across the water table, to then be transported though the groundwater system.

In the saturated zone, the advective-diffusive process is governed by two equations, namely the flow equation and the transport equation. Accurate, efficient solutions to the transport equation can be notoriously difficult to obtain, even when accurate solutions for the flow field are available. This situation is exacerbated by the large length to depth ratios common to most aquifers. Analytical solutions for the transport equation are readily available for infinite and semi-infinite flow domains, when the seepage velocity is constant (Bear, 1979; Hunt, 1983). However, in practical applications aquifers are of finite volume and irregular cross-section, and the seepage velocities are typically far from uniform.

Recently, the classical series method has been extended to cater for steady seepage problems, defined on irregular flow domains (Read & Volker, 1993; Read, 1995). As a consequence, an analytic solution can be obtained for the flow equation. Thus the flow field can be accurately and efficiently determined throughout the entire flow domain, and attention can be focussed on solving the transport equation. In addition to the potential solution ϕ, the conjugate stream function ψ is also immediately available. The potential and stream functions together form an orthogonal curvilinear coordinate system, in which the flow field is uniform. Hence the mass transport equation can be conformally transformed to a uniform flow domain, using potential and stream function coordinates.

In this paper, we provide analytic solutions for steady, isotropic diffusion. First, analytic solutions are obtained for the transport equation in the (ϕ, ψ) coordinate system, using analytic series methods. This solution is then transformed back to the original coordinate system, thus providing an analytic solution for the transport equation in the original seepage domain. In the next section, a formal mathematical description of the problem is given, together with details of the transformation process.

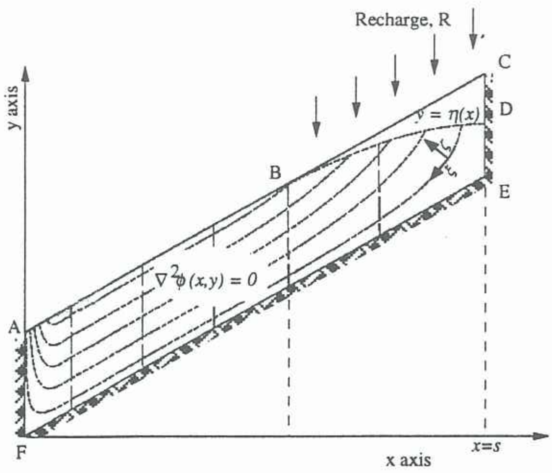

Figure 1. Schematic of the hillside seepage domain.

The series solution method is described next, followed by some representative results. Finally, the method and results are discussed.

PROBLEM FORMULATION

The two governing partial differential equations are the flow equation and the transformed transport equation. In order to formulate the problem mathematically, we make the following assumptions. First, the hillslope is homogeneous and isotropic, and subject to constant recharge at a rate sufficient to ensure the formation of a steady water table. Second, the transport processes in the aquifer have stabilised, so that any initial transients have decayed and steady state conditions prevail. Third, the solute is advected and diffused isotropically throughout the aquifer, after transmission through the water table. Finally, we assume that the solute is neutrally buoyant, so that the flow equation can be solved independent of the solute concentration. The governing equations and boundary conditions are given next.

Flow Equation

A schematic of the flow domain is presented in Figure 1. Seepage through porous media is governed by Darcy's law, and this reduces to Laplace's equation for the hydraulic head $\phi(x, y)$, when the hydraulic conductivity K is constant. In the saturated region ABDEF (Fig. 1), the governing equation for the flow field is

$$\nabla^2\phi(x, y) = 0. \tag{1}$$

On the impermeable boundaries AF, CE and FE, the Darcian flux across the boundary is zero. Along the bottom boundary $y = f^b(x)$, this condition becomes

$$\frac{\partial\phi}{\partial m} = \frac{1}{\sqrt{1+\dot{y}^2}}\left[\frac{\partial\phi}{\partial y} - \dot{y}\frac{\partial\phi}{\partial x}\right]_{y=f^b(x)} = 0. \tag{2}$$

Along the vertical side boundaries (at $x = 0, s$) the zero normal flux condition reduces to

$$\left[\frac{\partial\phi}{\partial y}\right]_{x=0,s} = 0. \tag{3}$$

The aquifer is subject to constant recharge $r = KR$, which leads to the formation of a steady water table BD, given by $y = \eta(x)$. Denoting the upper (saturated) flow boundary by $y^t(x)$, then

$$y^t(x) = \begin{cases} f^t(x) & , 0 \le x < \ell \\ \eta(x) & , \ell \le x \le s \end{cases} \tag{4}$$

Along the soil surface AB, the hydraulic head is equal to the elevation above an arbitrary datum. Denoting the upper (saturated) flow boundary by $y^t(x)$, this boundary condition becomes:

$$\phi^t(x) = \phi[x, y^t(x)] = y^t(x) \tag{5}$$

Another boundary condition applies, along the water table BD. Mass flux across the water table $y = \eta(x)$ is conserved, and this reduces to

$$R = \left[\frac{\partial\phi}{\partial y} - \dot{y}\frac{\partial\phi}{\partial x}\right]_{y=\eta(x)}. \tag{6}$$

This equation can be linearised, by first invoking the Cauchy-Riemann equations for the conjugate stream function $\psi(x, y)$ (Read & Volker, 1993), and then integrating with respect to x. Using the stream function, the free boundary condition (6) becomes

$$\psi[x, \eta(x)] = R(s - x), \tag{7}$$

where the arbitrary constant has been chosen so that the stream function is zero on the impermeable boundary AFEC.

The solution to the flow field can be obtained by first solving Laplace's equation (1) for the hydraulic head, subject to the boundary conditions (2), (3), (5) and (6) , in the context of a free boundary problem. The fluid (or pore) velocities (u, v) in the x and y directions are then given by

$$(u, v) = \left(-\frac{K}{\sigma}\frac{\partial\phi}{\partial x}, -\frac{K}{\sigma}\frac{\partial\phi}{\partial y}\right), \tag{8}$$

where σ is the (constant) porosity. Note that the stream function $\psi(x, y)$ and piezometric gradients $[\partial\phi(x, y)/x, \partial\phi(x, y)/y]$ are immediately available, once an analytic solution for $\phi(x, y)$ has been determined.

Transport Equation

The steady, isotropic advective-dispersive transport equation for the concentration $C(x, y)$ of solute in a (ξ, ζ) coordinate system, aligned parallel and perpendicular to the local direction of the fluid flow (i.e., parallel and perpendicular to the streamlines, in Figure 1), is given by (Hunt, 1983):

$$\frac{\partial}{\partial\xi}\left[D\frac{\partial C}{\partial\xi}\right] + \frac{\partial}{\partial\zeta}\left[D\frac{\partial C}{\partial\zeta}\right] - \bar{u}\frac{\partial C}{\partial\xi} = 0, \tag{9}$$

where D is the longitudinal and lateral diffusion coefficient, and $\bar{u} = \sqrt{u^2 + v^2}$ is the magnitude of the pore velocity. Note that in the general case D is not constant, and depends on $\bar{u}$.

In the (global) (x, y) coordinate system, the transport equation becomes (Hunt, 1983):

$$\frac{\partial}{\partial x}\left[D\frac{\partial C}{\partial x}\right] + \frac{\partial}{\partial y}\left[D\frac{\partial C}{\partial y}\right] - u\frac{\partial C}{\partial x} - v\frac{\partial C}{\partial y} = 0. \quad (10)$$

There will be no mass flux across the impermeable boundaries AF, FE and EC (Figure 1). As the pore velocity normal to an impermeable boundary is zero , the zero flux condition reduces to zero diffusion across the boundary (Bear, 1979). Along the impermeable base, the zero diffusion boundary condition becomes:

$$\frac{D}{\sqrt{1+\dot{y}^2}}\left[\frac{\partial C}{\partial y} - \dot{y}\frac{\partial C}{\partial x}\right]_{y=f^b(x)} = 0. \quad (11)$$

On the vertical side boundaries (at $x = 0, s$) the zero diffusion condition reduces to

$$\left[\frac{\partial C}{\partial x}\right]_{x=0,s} = 0. \quad (12)$$

Along the seepage face and water table, the concentration of solute is given by

$$C^t(x) = C[x, y^t(x)] = h^t(x), \quad (13)$$

where $h^t(x)$ is the measured or estimated concentration of solute.

Coordinate Transformation

In the (x, y) coordinate system, the isotropic transport equation involves the advection terms $u\partial C/\partial x$ and $v\partial C/\partial y$. These terms can be simplified, by transforming to a global coordinate system that is locally aligned to be parallel and perpendicular to the direction of fluid flow. The coordinate system (ϕ, ψ) satisfies this requirement and is related to the (x, y) coordinate system by a conformal transformation. The result is an orthogonal, curvilinear set of coordinates with scale factors $h_1 = h_2 = \bar{u}^{-1}$. The transport equation in these coordinates is (Bear, 1979 p. 246):

$$\frac{\partial}{\partial \phi}\left[D\frac{\partial C}{\partial \phi}\right] + \frac{\partial}{\partial \psi}\left[D\frac{\partial C}{\partial \psi}\right] - \frac{\partial C}{\partial \phi} = 0, \quad (14)$$

Note that (after simplification) the coefficient of the advection term in the transformed equation has been reduced to one. Assuming constant diffusivity D, this equation transforms to the Helmholtz equation

$$\bar{\nabla}^2 C - \alpha\frac{\partial C}{\partial \phi} = 0, \quad (15)$$

where $\alpha = D^{-1}$ and $\bar{\nabla}^2$ is the Laplacian operator, in the (ϕ, ψ) coordinate system.

The impermeable boundary AFEC (Fig. 1) in the (x, y) coordinate system transforms to the abscissa $\psi = 0$, in the (ϕ, ψ) coordinate system. The side and bottom boundary conditions (12) and (11) become

$$\left[\frac{\partial C}{\partial \psi}\right]_{\psi=0} = 0. \quad (16)$$

The top boundary $y = y^t(x)$ is transformed to $\psi = \psi^t(\phi)$, in the (ϕ, ψ) coordinate system. The top boundary condition (13) becomes

$$C^t(\phi) = C[\phi, \psi^t(\phi)] = \bar{h}^t(\phi), \quad (17)$$

where $\bar{h}^t(\phi)$ is the concentration along the top boundary ABD (Fig 1), transformed to the (ϕ, ψ) coordinate system.

The upper boundary $\psi^t(\phi)$ in the transformed domain can be determined pointwise, once an analytic solution to the flow domain has been generated. This boundary can then be represented using cubic splines or some other suitable interpolant.

ANALYTIC SERIES SOLUTIONS

The governing differential equations (1) and (15) for the flow field and mass transport respectively are both variations of the Helmholtz equation. Assuming homogeneous Neumann boundary conditions (at $\mu = 0, s$), a truncated analytic series solution for the Helmholtz equation $\nabla^2 f - \alpha\partial f/\partial\mu = 0$ is given by

$$f(\mu, \nu) \approx \sum_{n=1}^{N} A_n^f u_n(\mu, \nu) + B_n^f v_n(\mu, \nu) \quad (18)$$

where

$$u_n(\mu, \nu) = \cosh(\gamma_n \nu)\exp(\alpha\mu/2)\cos(\lambda_n \mu), \quad (19)$$

$$v_n(\mu, \nu) = \sinh(\gamma_n \nu)\exp(\alpha\mu/2)\cos(\lambda_n \mu), \quad (20)$$

and $\gamma_n = \sqrt{\alpha^2 + 4\lambda_n^2}/2$, $\lambda_n = (n-1)\pi/s$. Note that f is replaced with ϕ, α is set to zero and (μ, ν) becomes (x, y) for the flow equation, whereas f is replaced with C and (μ, ν) becomes (ϕ, ψ) for the transport equation. The series coefficients A_n^f andB_n^f are evaluated using the remaining boundary conditions.

Flow Solution

Assuming an estimate of the water table location is available, the free boundary problem is reduced to a sequence of known boundary problems, by iteratively improving the initial estimate. The recharge condition (7) is used as a cost function, to update the water table location at each step(full details of the iterative procedure are given in (Read, 1995)). Hence, the solution depends on solving the known boundary problem, at each step of the iterative process.

The bottom boundary condition for the flow equation can be expressed in terms of the series solution as

$$\sum_{n=1}^{N} A_n^\phi \bar{u}_n^b(x) + B_n^\phi \bar{v}_n^b(x) = 0, \quad (21)$$

where

$$\bar{u}_n^b(x) = \frac{1}{\sqrt{1+\dot{y}^2}}\left[\frac{\partial u_n}{\partial y} - \dot{y}\frac{\partial u_n}{\partial x}\right]_{y=f^b(x)}, \quad (22)$$

$$\bar{v}_n^b(x) = \frac{1}{\sqrt{1+\dot{y}^2}}\left[\frac{\partial v_n}{\partial y} - \dot{y}\frac{\partial v_n}{\partial x}\right]_{y=f^b(x)}. \quad (23)$$

Similarly, the potential condition along the upper boundary can be represented as

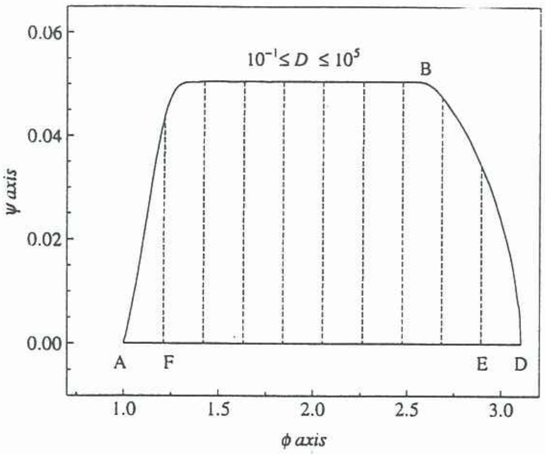

Figure 2. The transformed flow domain.

$$\sum_{n=1}^{N} A_n^{\phi} u_n^t(x) + B_n^{\phi} v_n^t(x) = y^t(x), \tag{24}$$

where $u_n^t(x) = u_n[x, y^t(x)]$ and $v^t(x) = v_n[x, y^t(x)]$. The series coefficients A_n^{ϕ} and B_n^{ϕ} can be estimated, using the principle of eigenfunction expansions, applied to non-orthogonal basis functions (Read, 1995).

Transport Solution

The bottom boundary condition for the flow equation can be satisfied exactly in the transformed domain, as the bottom boundary is horizontal. Hence the series solution for the transport equation becomes

$$C(\phi, \psi) \approx \sum_{n=1}^{N} B_n^C v_n(\phi, \psi), \tag{25}$$

The top boundary condition (17) can be expressed as

$$\sum_{n=1}^{N} B_n^C v_n^t(\phi) = \bar{h}^t(\phi), \tag{26}$$

where $v_n^t(\phi) = v_n[\phi, \psi^t(\phi)]$. As noted previously, the series coefficients B_n^C can be estimated using techniques based on the principle of eigenfunction expansions.

RESULTS

The flow domain depicted in Figure 1 has been chosen to demonstrate the solution technique, as it is typical of hillside seepage regions. The basal and surface slopes are 5% combined with a length to depth ratio (i.e., s) of 50:1. Recharge arrives at the soil surface at a rate of $R = 10^{-3}$. A powerful feature of the analytic series method is that exact bounds on the truncation error are immediately available, and can be determined by examining the boundary errors. For the specified flow geometry, between 10 and 15 terms in the series were sufficient to ensure that the root mean square (rms) errors were $\approx 10^{-3} - 10^{-4}$. Once the potential solution has been generated, the stream function is also available. In the (ϕ, ψ) coordinate system, the flow region transforms to the domain shown in Figure 2.

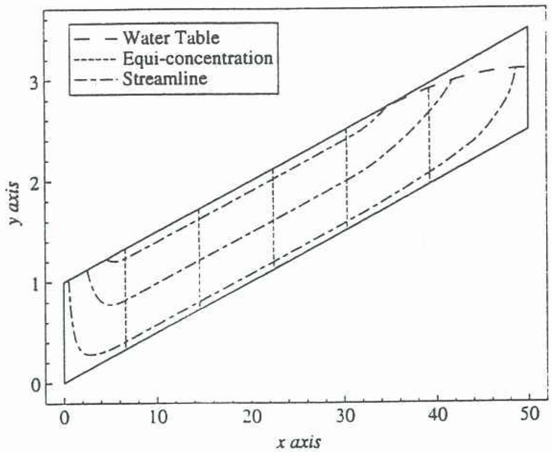

Figure 3. Concentration Solutions.

The transport equation can be solved in the (ϕ, ψ) system, once the diffusivity D and the concentration profile along $y^t(x)$ has been specified. In the general case, D is proportional to the velocity. Assuming that the porosity is in the range 10% – 30%, and the velocity is approximately constant, then $10^{-1} \leq D \leq 1$. However, solutions have been obtained with diffusivity up to five orders of magnitude larger than the upper bound, to demonstrate the relative insensitivity of the solution to the value of D (Fig. 2). Similarly, a variety of concentration profiles $C^t(x)$ were chosen, each increasing from 0 at $x = 0$ to 1 along the upstream region of the water table (e.g., $C^t(x) = (x/s)^n$ where $n = 1, 2, 3$). Surprisingly, the solutions to the transport equation for these different concentration profiles and $10^{-1} \leq D \leq 10^5$ are almost indistinguishable from one another, and the contours of concentration are (approximately) aligned along the equi-potentials.

CONCLUSIONS

The steady, isotropic advection-diffusion equation has a simplified form in the (ϕ, ψ) coordinates. The availability of an analytic solution to the flow equation (via the analytic series method) means that it is possible to do this transformation. Hence, analytic solutions to the transport equation can also be found, in the transformed domain. The solutions presented in this paper are somewhat surprising, as the influence of the diffusion term appears to be negligible. This is possibly due to the large aspect ratio of the flow domain. The authors are currently investigating this question.

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