Twelfth Australasian Fluid Mechanics Conference
The University of Sydney, Australia 1995

SPATIALLY-PERIODIC STEADY SOLUTIONS OF THE NAVIER-STOKES EQUATIONS WITH THE ABC-FORCE

Olga M.Podvigina[1]
School of Mathematics and Statistics
University of Sydney
New South Wales
Australia

ABSTRACT

An ABC-flow is a steady solution of the Navier-Stokes equation with periodic boundary conditions and a corresponding force. Three symmetric steady solutions different from the original one are found numerically for $20 \leq R \leq 200$. Their behaviour supports the hypothesis that this family of solutions persists for $R \to \infty$.

Steady solutions of the Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{v}) - \nabla p + \frac{1}{R}\Delta \mathbf{v} + \frac{\mathbf{F}}{R}$$

with the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ and periodic boundary conditions are investigated numerically in the interval of Reynolds number R from 20 to 200. The ABC flow

$$\mathbf{u}_{ABC} = (A \sin kx_3 + C \cos kx_2,\ B \sin kx_1 + A \cos kx_3,\ C \sin kx_2 + B \cos kx_1)$$

is a steady solution of the equation for the selected force $\mathbf{F} = k^2 \mathbf{u}_{ABC}$. The present study is restricted to the case $A = B = C = 1,\ k = 1$.

As a solution of the Navier-Stokes equation $\mathbf{u}_{ABC}$ becomes linearly unstable at $R \approx 13.05$ [Galloway & Frisch 1987]. As R grows from 13 to 50 the time-dependent solution undergoes 11 bifurcations [Podvigina (Zheligovsky) & Pouquet 1993,1994] resulting in an increasing complexity of the flow. The computations at $13 \leq R \leq 25$ revealed existence of three steady solutions different from $\mathbf{u}_{ABC}$. These steady solutions are mutually symmetric, the symmetries arising due to the symmetries admitted by the 1:1:1 ABC-force. They loose linear stability at $R \approx 14.0$ via the Hopf bifurcation.

Together with the original ABC-flow these solutions play the most important rôle in the dynamics of the system. At $R \geq 14.7$ a time-dependent flow experiences chaotic jumps between $\mathbf{u}_{ABC}$ and the secondary steady flows, with one steady flow involved in any typical evolution at $14.7 \leq R \leq 18$, and with all the three visited within an individual evolution at $20 \leq R \leq 25$. The temporal record of the energy during the evolution consists of plateaux corresponding to the steady solutions separated by intervals of noisy behaviour. As R grows the plateaux become shorter - from 2000 turnover times at $R = 15$ to total disappearance; the noisy phases' lengths increase. At $R = 30$ the temporal behaviour becomes completely unstructured.

The significant rôle of the steady solutions in the dynamics of the hydrodynamic system, on one hand, and the fact that their mere existence at large Reynolds numbers is questionable [Arnold, 1992], on the other, makes them an important object

[1] E-mail: podvigina_o@maths.su.oz.au

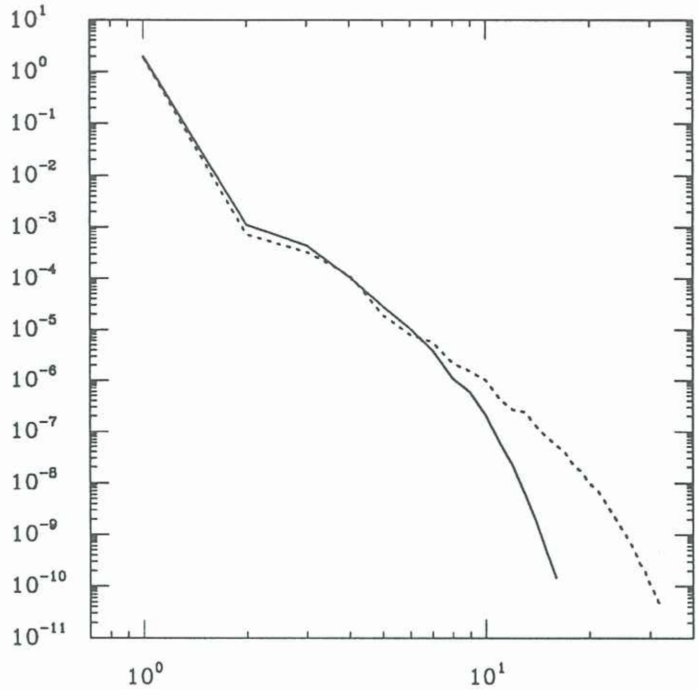

Figure 1. The energy spectra of the solutions at $R = 70$ (solid line) and $R = 200$ (dotted line).
Horisontal axis: wavenumber K, vertical axis: $\Sigma_{K \leq |\mathbf{k}| < K+1} |\mathbf{v_k}|^2$.

to study. In the present research we establish numerically the presence of the secondary solutions at $20 \leq R \leq 200$ and investigate their properties.

The steady solutions were obtained for $20 \leq R \leq 100$ step 10 and for $100 \leq R \leq 200$ step 20 by a new specially designed numerical method [Podvigina & Zheligovsky, 1995]. The pseudospectral method was applied, with the solutions represented as Fourier series

$$\mathbf{v} = \sum_{\mathbf{k}} \mathbf{v_k} e^{i\mathbf{kx}}.$$

Computations were made with the resolution 32^3 Fourier harmonics for $R \leq 70$ and 64^3 at higher R. The energy spectra of the solutions at $R = 70$ and 200 are represented on Figure 1 showing that the resolution is adequate.

Present results support the hypothesis that the considered family of the steady solutions exists at large R.

The new steady solution (speaking about one in the family of three) is close to an ABC-flow with $A = C \neq B$ and A, B and C depending on R. The discrepancy between the steady solution and the ABC-flow is at least by two orders of magnitude smaller in energy than the flows themselves. Figure 2 illustrates the behaviour of the components of the solution with the unit wavevector. The wavevector 1 components of any periodic solenoidal field admitting the symmetries of the steady flow under consideration can be represented as a sum of an ABC flow $\mathbf{u}_{ABC}$ with $k = 1$ and $A = C$ and of an ABC flow $\mathbf{u}_{A'B'C'}$ with $k = -1$, $A' = -C'$ and $B' = 0$. Figure 2 shows the behaviour of $\mathrm{Im}(v^1_{0,0,-1}) = (A - A')/2$, $\mathrm{Re}(v^2_{0,0,1}) = (A + A')/2$ and $\mathrm{Re}(v^3_{1,0,0}) = (B + B')/2$. Thus Figure 2 confirms that the wavevector 1 component of the solution is close to an ABC flow $\mathbf{u}_{ABC}$ with $k = 1$ and $A = C$. However, it also shows that at large Reynolds numbers $\mathrm{Re}(v^2_{0,0,1}) - \mathrm{Im}(v^1_{0,0,-1}) = A'$ does not tend to zero, and hence the solution does not tend to the ABC flow.

The Fourier coefficients are changing significantly at the interval $20 \leq R \leq 70$, but practically do not display considerable variation when the Reynolds number grows further up to $R = 200$.

Figure 3 represents $\mathrm{Im}(v^3_{1,1,0})$, $\mathrm{Re}(v^2_{1,1,0})$ and $\mathrm{Im}(v^1_{-1,-1,-1})$ (these coefficients vanish in any ABC-flow). As the Reynolds number grows the first two of them decrease monotonously (at $R = 200$ they are by an order of magnitude smaller than at $R = 20$). The behaviour of $\mathrm{Im}(v^1_{-1,-1,-1})$ is more complicated; if it tends to a limit when $R \to \infty$, this limit is supposedly non-zero.

The steady solutions are currently being computed at higher Reynolds numbers. The results will provide more information on existence and the limit behaviour of the Fourier coefficients of the solutions and will hopefully enable to construct an analytical asymptotic decomposition for large R.

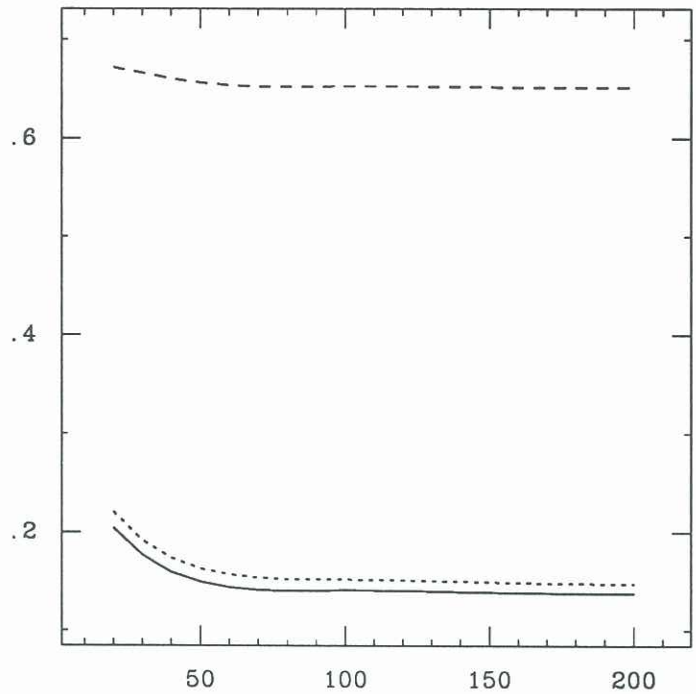

Figure 2. Wavevector 1 components of the steady flows. Horisontal axis: Reynolds number.
Vertical axis: $\mathrm{Im}(v^1_{\bar{0},0,-1})$ (solid line), $\mathrm{Re}(v^2_{\bar{0},0,1})$ (dotted line), $\mathrm{Re}(v^3_{\bar{1},0,0})$ (dashed line).

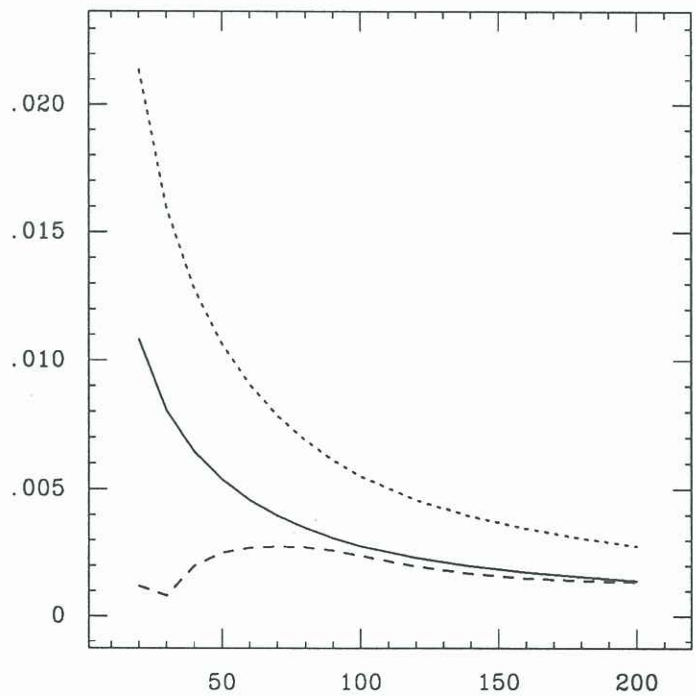

Figure 3. Components of the steady flows. Horisontal axis: Reynolds number.
Vertical axis: $\mathrm{Re}(v^2_{\bar{1},1,0})$ (solid line), $\mathrm{Im}(v^3_{\bar{1},1,0})$ (dotted line), $\mathrm{Im}(v^1_{-1,-1,-1})$ (dashed line).

ACKNOWLEDGMENTS

Computations were made on the CM-5 at the Sydney Regional Centre for Parallel Computing (University of New South Wales, Australia). I am grateful to V.Zheligovsky for helpful conversations and comments.

REFERENCES

Arnold V.I., 1992, private communication.

Galloway, D.J., and Frisch, U., 1987, "A note on the stability of a family of space–periodic Beltrami flows", *Journal of Fluid Mechanics*, Vol. 180, pp. 557-564.

Podvigina, O., and Pouquet, A., 1994, "On the non–linear stability of the 1:1:1 ABC flow", *Physica D*, Vol. 75, pp. 471-508.

Podvigina, O.M., and Zheligovsky, V.A., 1995, "An iterative method for the numerical solution of large systems of equations based on the properties of the Chebyshev polynomials.", *Report 95-20*, School of Mathematics and Statistics, The University of Sydney.

Zheligovsky (Podvigina), O., and Pouquet, A., 1993, "Hydrodynamic Stability of the ABC Flow", *Solar and Planetary Dynamos* (Eds. M.R.E. Proctor, P.C. Matthews and A.M. Rucklidge), pp. 347-354, Cambridge Univ. Press.