

THE THREE-DIMENSIONAL MEAN VELOCITY FOR STRATIFIED TURBULENT FLOWS:- THE TEMPORAL-RESIDUAL-MEAN VELOCITY

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ABSTRACT

When temporally averaging the density conservation equation over unresolved (eddy) motions in z -coordinates for a stratified fluid such as the atmosphere or the ocean, eddy fluxes arise due to the correlation between the unresolved perturbation velocity and density fields. Even when the instantaneous motions are adiabatic and when the flow is statistically steady, the divergence of these eddy fluxes is non-zero and leads to unwanted "diabatic" or "diapycnal" advection by the mean Eulerian three-dimensional flow field. A newly discovered three-dimensional mean velocity field is shown to not suffer this problem. By analogy with the residual mean circulation that arises when zonally averaging, the new three-dimensional velocity is called the temporal-residual-mean velocity, or TRM velocity for short. It is this velocity that must be used in eddyless models of three-dimensional turbulent stratified fluids such as the ocean or atmosphere.

INTRODUCTION

When the conservation equations are averaged in density coordinates, the only diapycnal velocity that appears is due to diapycnal mixing processes that are present in the un-averaged, instantaneous state. If the instantaneous flow field is always adiabatic then the averaged flow field is also adiabatic. This is true both for the zonal averaging operator (see section 9.4.3 of Andrews et al (1987)) when the resulting averaged flow is two-dimensional, and for the temporal averaging operator (see, for example, section 1 of Gent et al (1995)) when the flow field is three-dimensional. In both cases, the isopycnally averaged velocity is directed in the averaged density surface. The work reported here enables this same adiabatic property to be obtained for a temporally averaged velocity vector in the z -coordinate reference frame.

Because of the importance of diapycnal transport for the overturning circulation of the ocean and also for the density and tracer properties of deep water, it is important to understand the processes that lead to diapycnal advection in reality and also in ocean models. In layered (isopycnal) ocean models it is now possible to ensure that the diapycnal diffusion and advection of properties occurs only by processes that one explicitly places in the model (McDougall and Dewar, 1996). The same is not automatically true for models that are cast with respect to z -coordinates. The Reynolds decomposition of properties into mean and perturbation quantities shows that the advection of mean density by the Eulerian-mean flow is not zero but rather is equal to the divergence of the three-dimensional eddy flux of density. In addition, as a practical matter in eddyless ocean models, it is necessary to include exactly horizontal diffusion and this causes unwanted and unphysical "fictitious" diapycnal density fluxes (Veronis, (1975), McDougall and Church (1986)) that adversely affects the deep water properties (Danabasoglu et al (1994), Hirst and McDougall, 1996a) and the meridional oceanic heat flux (Hirst and McDougall, 1996b). It is argued here that the numerical techniques that have been pioneered by Gent and McWilliams (1990) and Gent et al (1995) are really techniques for implementing the temporal-residual-mean circulation, and that this is the reason why the flow with respect to the total velocity vector in these works can be taken to be adiabatic.

THE TRM STREAMFUNCTION

The instantaneous conservation equation for neutral density is $\gamma_t + \mathbf{V} \cdot \nabla_H \gamma + w \gamma_z = Q$ where Q represents the unresolved production (due to molecular mixing processes and radiative heat flux divergence) and the horizontal and vertical velocities are \mathbf{V} and w respectively. Neutral density is the oceanographic density variable that correctly accounts for the complicated compressible nature of seawater:— it is defined by Jackett and McDougall (1996), and it corresponds to potential temperature in the atmosphere. Each term in the instantaneous density conservation equation is now Reynolds-decomposed into a low-passed temporal average value and a deviation from this running mean, leading to the following standard equations for the conservation of the mean density and the density variance,

$$\bar{\gamma}_t + \bar{\mathbf{V}} \cdot \nabla_H \bar{\gamma} + \bar{w} \bar{\gamma}_z = \bar{Q} - \nabla_H \cdot (\bar{\mathbf{V}}' \gamma') - (\bar{w}' \gamma')_z \quad (1)$$

and

$$\bar{\phi}_t + \bar{\mathbf{V}} \cdot \nabla_H \bar{\phi} + \bar{w} \bar{\phi}_z = -\left\{ \bar{\mathbf{V}}' \gamma' \cdot \nabla_H \bar{\gamma} + \bar{w}' \gamma' \bar{\gamma}_z \right\} + \bar{Q}' \gamma' + O(\alpha^3), \quad (2)$$

where $\bar{\phi} = \frac{1}{2} \overline{(\gamma')^2}$. Here α stands for a perturbation amplitude so that in (2) there are additional terms that are of cubic or higher powers in perturbation amplitude. Note that all of the perturbation quantities here are evaluated at constant height.

In the zonal mean case of Andrews and McIntyre (1976), the residual mean northward velocity was found to be the Eulerian mean velocity plus the vertical derivative of the scalar streamfunction, $-\bar{v}' \theta' / \bar{\theta}_z$, where the overbar is the zonal averaging operator and v' is the deviation of the northward velocity from this zonal average. By analogy with this zonal-mean case, we might assume that the streamfunction for the temporally averaged situation would be the horizontal eddy density flux divided by the mean vertical density gradient. This turns out to not be sufficient. Instead, the following mean advection field is chosen

$$\bar{\mathbf{V}}^\# = \bar{\mathbf{V}} + \Psi_z \quad \text{and} \quad \bar{w}^\# = \bar{w} - \nabla_H \cdot \Psi \quad (3)$$

where the extra vector streamfunction is two-dimensional with both eastward and northward components and is taken to be

$$\Psi = -\frac{\bar{\mathbf{V}}' \gamma'}{\bar{\gamma}_z} + \mathbf{A}, \quad (4)$$

because it is recognized that the analogy with the zonal mean case may not be complete and an additional component, \mathbf{A} , to the streamfunction may be required. The mean density equation, (1), is now written with respect to the TRM advection velocity, and the density variance equation, (2), is used to eliminate the term, $-(\bar{w}' \gamma')_z$, yielding

$$\bar{\gamma}_t + \bar{\mathbf{V}}^\# \cdot \nabla_H \bar{\gamma} + \bar{w}^\# \bar{\gamma}_z = \bar{Q} - \left(\frac{\bar{Q}' \gamma'}{\bar{\gamma}_z} \right)_z + \left(\frac{\bar{\phi}_t}{\bar{\gamma}_z} \right)_z + \mathbf{A}_z \cdot \nabla_H \bar{\gamma} - \bar{\gamma}_z \nabla_H \cdot \mathbf{A} + \left(\frac{\bar{\mathbf{V}} \cdot \nabla_H \bar{\phi} + \bar{w} \bar{\phi}_z}{\bar{\gamma}_z} \right)_z + O(\alpha^3). \quad (5)$$

Bearing in mind maps of mesoscale eddy variability derived from satellite altimetry and assuming that maps of density variance at depth in the ocean have a similar appearance, we recognize that the advection of density variance by the mean flow, $\bar{\mathbf{V}} \cdot \nabla_H \bar{\phi} + \bar{w} \bar{\phi}_z$, is not small, and a scale analysis of the terms in (5) shows that the term due to this advection of density variance is as large as any other term in the equation. It is the role of the extra streamfunction, \mathbf{A} , to negate the influence of this term. The derivation of the required expression for \mathbf{A} has been performed by McDougall and McIntosh (1996a) where it is shown that the total streamfunction becomes

$$\Psi = -\frac{\bar{\mathbf{V}}' \gamma'}{\bar{\gamma}_z} + \frac{1}{\bar{\gamma}_z} \left(\frac{\bar{\mathbf{V}} \bar{\phi}}{\bar{\gamma}_z} \right)_z \quad (6)$$

and the mean density conservation equation (5) with respect to this TRM velocity is

$$\bar{\gamma}_t + \bar{\mathbf{V}}^\# \cdot \nabla_H \bar{\gamma} + \bar{w}^\# \bar{\gamma}_z = \bar{Q} - \left(\frac{\bar{Q}' \gamma'}{\bar{\gamma}_z} \right)_z + \left(\frac{1}{\bar{\gamma}_z} \left[\frac{\bar{\phi} \bar{Q}}{\bar{\gamma}_z} \right] \right)_z + \left(\frac{\bar{\phi}_t}{\bar{\gamma}_z} \right)_z - \left(\frac{1}{\bar{\gamma}_z} \left[\frac{\bar{\phi} \bar{\gamma}_t}{\bar{\gamma}_z} \right] \right)_z + O(\alpha^3). \quad (7)$$

For steady flow ($\bar{\gamma}_t = \bar{\phi}_t = 0$), and for adiabatic motions, ($\bar{Q} = Q' = 0$), this conservation equation becomes $\bar{\mathbf{V}}^\# \cdot \nabla_H \bar{\gamma} + \bar{w}^\# \bar{\gamma}_z = 0$, showing that the mean advection through $\bar{\gamma}$ surfaces is zero:— the TRM velocity field, $\bar{\mathbf{V}}^\# + \mathbf{k} \bar{w}^\#$, has absorbed all the effects of the adiabatic stirring accomplished by all types of mesoscale motions. This contrasts sharply with the usual Eulerian-averaged equation, (1), which contains the divergence of the three-dimensional flux vector.

In the case of zonal averaging, the mean northward and vertical velocities are of second order in perturbation quantities and so the advection of density variance, $\bar{v}\bar{\phi}_y + \bar{w}\bar{\phi}_z$, does not appear at leading order in the variance equation. However, in the case of temporal averaging, the advection of density variance in (2) appears at leading order in perturbation quantities and so cannot be ignored. This considerably complicates the derivation. In particular, Holton (1981) was able to conclude that for adiabatic, small amplitude disturbances in a steady state, the flux of potential temperature due to the zonal perturbations was parallel to the isolines of $\bar{\theta}$ in the y - z plane. In the temporal averaging situation, no such simplification is possible. Because the mean flow velocities in (2) are of zeroth order in perturbation amplitude, that is, $O(\alpha^0)$, the three-dimensional eddy flux of neutral density will not be aligned in the neutral density surface, that is, $\bar{v}'\gamma' \cdot \nabla_H \bar{\gamma} + \bar{w}'\gamma' \bar{\gamma}_z \neq 0$, even at leading order and for a steady state.

ON THE AVERAGING OF SCALAR FIELDS

Since the TRM velocity is the correct velocity for the advection of the mean density field in three dimensions, the question arises of what type of tracer variable is advected by the TRM velocity. There are essentially three choices, (i) the Eulerian averaged tracer value, being the value averaged at constant depth (ii) the tracer averaged on density surfaces, and (iii) the thickness-weighted tracer value, evaluated as the average value between a pair of density surfaces. It is shown by McDougall and McIntosh (1996a) that it is option (ii), the value averaged on density surfaces that is advected by the TRM velocity field. This result may seem quite surprising since the whole derivation is based on a Reynolds decomposition in z -coordinates. This result is confirmed by the z -coordinate form of the isopycnal conservation equations that is developed in McDougall and McIntosh (1996b) and this provides the physical explanation. There it is shown that the TRM approach to the conservation of density and tracers actually achieves the conservation of volume, density and tracers in an isopycnal framework, but has these layered conservation statements expressed in terms of properties measured in z -coordinates.

It has recently been shown by Lozier et al (1994) that averaging potential temperature and salinity at fixed height can create water masses that do not exist in the real ocean. This effect is due to the curvature of the salinity-potential temperature curve combined with the vertical heaving of the water column caused by meso-scale eddies and is especially pronounced when the data is averaged horizontally over large distances such as in the Levitus (1982) atlas. The theoretical work of McDougall and McIntosh (1996b) provides further justification to the thesis of Lozier et al (1994):- namely that properties should be averaged along density surfaces so that known oceanic water masses retain their integrity. This theoretical work shows that not only is this the correct form of averaging for preserving known water-mass properties, but also it is the correct way of averaging data in order to (i) compare with a z -coordinate ocean model, (ii) to assimilate such data into an ocean model, and (iii) to use in an inverse model. Just because one's model (whether a forward or an inverse model) is formulated in z -coordinates does not mean that the tracer values should be averaged in z -coordinates. To do so will introduce errors. Rather, the correct procedure is to average the tracers on the neutral density surface whose neutral density is equal to the Eulerian-averaged neutral density at the height in question.

In the case of zonal averaging, it is immaterial at leading order in perturbation quantities whether tracers are averaged at constant height or at constant potential temperature, so that the distinction has not previously arisen.

DISCUSSION

A new three-dimensional mean velocity field is presented when the averaging operator is temporal and the coordinate frame is fixed in space, such as the common z -coordinate system. This velocity field has the property that when the mean flow is steady and for disturbances that are instantaneously adiabatic, the new velocity field is adiabatic, that is, the advection of mean density by the new velocity has no forcing terms. This is the same desirable property that is achieved by the two-dimensional residual circulation in the atmosphere (Andrews and McIntyre (1976) and in the ocean (McIntosh and McDougall (1996)) under the zonal averaging operator. Because of this analogy with the zonal residual mean circulation, we call the new velocity vector the temporal-residual-mean velocity, often abbreviated to the TRM velocity. The adiabatic nature of the mean velocity arises trivially in density coordinates under the same physical assumptions, but to date, there has not been the corresponding expression for the appropriate mean velocity in z -coordinates.

Prognostic eddyless numerical models of the ocean circulation need to be able to specify the amount of diapycnal mixing and advection independently of the horizontal stirring achieved by mesoscale eddy activity, and the TRM velocity is the first mean velocity that achieves this. Since the work of Gent and McWilliams (1990) and Gent et al (1995) it is now possible to include in ocean models an extra advection based on a streamfunction of one's choosing. In the previous work in this area it has been assumed that the extra streamfunction had the role of providing the missing "bolus velocity" to the z -coordinate model:- the bolus velocity being due to the correlations between the instantaneous horizontal velocity and the instantaneous thickness between a pair of density surfaces. The work of McDougall and McIntosh (1996b) has analysed the

conservation equations in density coordinates and has shown that the extra streamfunction that is needed in z -coordinates must provide more than just the bolus velocity. It must also provide (i) the difference between the mean velocity measured on a density surface and that measured at constant height, and (ii) another term that arises because the average height of the Eulerian-averaged neutral density surface is not equal to the height at which the original average was performed. The TRM streamfunction provides all three of these additional velocities. Hence this work provides the theoretical support for the general approach of using a vector streamfunction that has been pioneered by Gent and McWilliams (1990) and by Gent et al (1995). The actual numerical techniques advocated by Gent et al (1995) need very few changes. Rather, what is needed is a reinterpretation of the various velocity vectors. Subject to this re-interpretation of the Gent et al (1995) velocity vectors, and the fact that the model's tracer values are actually the isopycnally averaged tracer values, the present theory provides strong theoretical support for this way of including the effects of mesoscale eddies in z -coordinate models of stratified turbulent fluids.

When considering the conservation equation for a tracer in the ocean or atmosphere, McDougall and McIntosh (1996b) have shown that the correct form of averaging for the tracer is that along density surfaces. Just because one's model (whether a forward or an inverse model) is formulated in z -coordinates does not mean that the tracer values should be averaged in z -coordinates. This surprising result has important implications for the averaging of data for the purpose of (i) comparing with ocean models, (ii) to assimilate such data into ocean models, (iii) to use in an inverse model and (iv) for the formation of atlases of averaged properties such as is planned as a product of the World Ocean Circulation Experiment. While we may not have realised it to date, ocean models that have been used in oceanography for the past twenty five years have in fact carried the isopycnally-averaged tracer values. When a parameterization of the TRM circulation such as that of Gent and McWilliams (1990) is used in such models, this realization can be used to good effect to simulate water masses quite accurately (Hirst and McDougall (1996)). This means that the models inherently have a greater ability to simulate the ocean than has been realised to date. If it were not for this property, errors of up to 1°C would be inevitable in places such as the Gulf Stream (Lozier et al (1994)).

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