Twelfth Australasian Fluid Mechanics Conference
The University of Sydney, Australia 1995



# VELOCITY GRADIENT INVARIANT EVOLUTION FROM A LINEAR DIFFUSION MODEL

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## ABSTRACT

The velocity gradient tensor equation is modelled in probability phase space with a linear approximation for the diffusion term and a diagonal isotropic model for the pressure Hessian. This formulation results in a closed equation system for the velocity gradient invariant evolution. The resultant local topology for the flow kinematics is similar to known results of numerical experiments. In particular, the well established tendency for the intermediate strain rate eigenvalue to be positive in isotropic homogeneous turbulence is predicted by this model.

## INTRODUCTION

Velocity gradients contain information on the rates of rotation, stretching and angular deformation of infinitesimal material lines, surfaces and volumes. These are distinctive fluid flow mechanisms, which are essential in a broad range of important phenomena such as the propagation of material and non material surfaces. Vortex stretching by velocity gradients is also fundamental to the investigation of vorticity dynamics and turbulent kinetic energy generation.

The internal intermittency of turbulence is also connected to strongly non-Gaussian velocity gradient statistics. This statistics implies high probability of large values of the velocity derivatives or, equivalently, a spotty distribution of regions with large amplitude of velocity gradients.

Villiefosse (1982) investigated the inviscid velocity gradient transport equation by neglecting the off-diagonal elements of the pressure Hessian. With these crude approximations the tendency of the vorticity to align with the intermediate strain rate eigenvector and its local increase without limit in finite time is explained. Chong *et al.* (1990) classify the topology of fluid motions in terms of the three invariants of the velocity gradient tensor, which completely determine the dynamic system associated to a local origin, which in a Lagrangian frame is a critical point. Analogous invariants can be defined for the strain-rate tensor and the rate of rotation tensor associated with the vorticity. Chen *et al.* (1990) have studied the dissipative scales of mixing layers in compressible and incompressible cases in terms of the three invariants using DNS (Direct Numerical Simulation) and suggested a phenomenological relation between the second and the third. Cantwell (1992) has found an analytical solution to Villiefosse model equation in terms of Jacobian elliptic functions and obtained the asymptotic behaviour of vorticity and rate of strain, using the invariants of this last tensor in order to explore the small scale flow structure. Soria *et al.* (1994) have studied the characteristics of plane turbulent mixing layers with different initial conditions using DNS and have analysed the velocity gradient tensor and strain-rate topologies for each case, finding highly localised regions in the invariant scatter-plots.

## MODELLED EQUATION FOR THE VELOCITY GRADIENT TENSOR

The evolution equations for the velocity gradient tensor components $g_{ij} = \partial u_i(x,t)/\partial x_j$ in a Lagrangian frame following the fluid particle in homogeneous isotropic turbulence with constant density are

$$\frac{dg_{ij}}{dt} = -g_{ik}g_{kj} - \frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_j} + \nu\frac{\partial^2 g_{ij}}{\partial x_k \partial x_k} \qquad (1)$$

The conditional time derivative of $g_{ij}$ is defined by the following conditional average

$$\left\langle \frac{dg_{ij}}{dt} \mid g_{ij}(x,t) = G_{ij} \right\rangle = \frac{dG_{ij}}{dt} \tag{2}$$

Where $G_{ij}$ is the associated variable in the corresponding probabilistic phase space to the physical variable $g_{ij}$.

Taking the conditional averages of the terms in eq. (1), it becomes

$$\frac{dG_{ij}}{dt} = -G_{ik}G_{kj} + \left\langle -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_j} \mid G_{ij} \right\rangle + \left\langle \nu \frac{\partial^2 g_{ij}}{\partial x_k \partial x_k} \mid G_{ij} \right\rangle \tag{3}$$

The first term on the right hand side of eq. (3) is the velocity gradient self-straining/rotation. The two conditional averages correspond to the pressure Hessian and viscous diffusion. These need to be modelled in order to close this equation in terms of the $G_{ij}$ variables.

The pressure term is modelled by a local isotropy approximation (Dopazo *et al.* (1993)). The result is analogous to Villiefosse (1982)

$$\left\langle -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_j} \mid G_{ij} \right\rangle = \frac{1}{3}\delta_{ij} G_{lm} G_{ml} \tag{4}$$

For the diffusion term a linear model, Linear Mean Square Estimation (LMSE), is used (Dopazo *et al.* (1993))

$$\left\langle \nu \frac{\partial^2 g_{ij}}{\partial x_k \partial x_k} \mid G_{ij} \right\rangle = -\omega_0 G_{ij} \tag{5}$$

Where the constant $\omega_0$ is a "diffusion frequency", the inverse of a presumed diffusion characteristic time of the velocity spatial derivatives.

Substituting both models in eq. (3) the evolution of the components of the velocity gradient tensor in probabilistic phase space is given as

$$\frac{dG_{ij}}{dt} = -G_{ik}G_{kj} + \frac{1}{3}\delta_{ij}G_{lm}G_{ml} - \omega_0 G_{ij} \tag{6}$$

This resulting set of deterministic equations constitute a dynamic system of eight independent variables because of the incompressibility condition $G_{ii} = 0$. Furthermore, the model preserves zero mean for each component, ($\langle G_{ij} \rangle = 0$), for all time.

The divergence of the velocity field, $dG_{ij}/dt \equiv \dot{G}_{ij}$, in phase space results in

$$\frac{\partial \dot{G}_{ij}}{\partial G_{ij}} = -9\omega_0 < 0, \tag{7}$$

i.e. a negative, constant value of $-9\omega_0$ is obtained. Thus the system (6) is dissipative, with constant contraction of volume with time in phase space.

## EVOLUTION OF THE INVARIANTS AND RESULTS

The characteristic equation of a second order tensor $G_{ij}$ is

$$\lambda^3 + P\lambda^2 + Q\lambda + R = 0 \tag{8}$$

where $\lambda$ stands for the eigenvalues (real or complex) of the tensor and $P$, $Q$, $R$ are the invariants, defined by

$$P = -G_{ii}$$

$$Q = \frac{1}{2}(P^2 - G_{ij}G_{ji}) \tag{9}$$

$$R = \frac{1}{3}(-P^3 + 3PQ - G_{ij}G_{jk}G_{ki})$$

Here $G_{ij}$ is the velocity gradient tensor. Its invariants determine, through the eigenvalues, the local topology of the motions in the flow near the origin considered as a critical point (see Chong *et al.* (1990) for details). In the incompressible case $P$ is zero, and $Q$ and $R$ reduce respectively to the traces of the square and cube of the tensor with adequate factors.

The real or complex character of the eigenvalues is given by the sign of the discriminant $D$ defined, when $P = 0$, as

$$D = \frac{R^2}{4} + \frac{Q^3}{27}. \tag{10}$$

In the general solution of the cubic algebraic equation (8) the three roots are real when $D \leq 0$. When $D > 0$ two are conjugated complex and one is real. Then, for the incompressible case, the null discriminant curve $D = 0$ divides the $QR$ plane in two regions: One below the curve with real eigenvalues and other above the curve with one real and two conjugated complex ones.

Taking the time derivatives of $Q$ and $R$ in definition (9), and substituting the proposed model (6) for the time derivative of $G_{ij}$ , yields a closed, self-consistent system of equations for the evolution of the invariants.

$$\dot{Q} = -3R - 2\omega_0 Q$$

$$\dot{R} = \frac{2}{3}Q^2 - 3\omega_0 R \tag{11}$$

This dynamic system is non-linear due to the $Q^2$ term in the equation for $\dot{R}$

The discriminant evolution can be found by taking the time derivative of eq. (10) and replacing the invariant derivatives (11). The solution of the resulting equation is

$$D(t) = D(t=0)\, exp\{-6\omega_0 t\} \tag{12}$$

This result implies that there is no sign change in the discriminant. Therefore, the eigenvalue character (real or complex) of the velocity gradient is the same all the time.

The system (11) has two fixed points ($\dot{Q} = 0$, $\dot{R} = 0$):

$$P_1 \equiv (Q_1 = 0\,,\ R_1 = 0)$$

$$P_2 \equiv (Q_2 = -3\omega_0^2\,,\ R_2 = 2\omega_0^3)$$

both of them lying on the curve $D = 0$. The associated Jacobian matrix has in $P_1$ two negative eigenvalues, and in $P_2$ one positive and one negative. This implies that the former is a stable node and the later is a saddle. Figure 1 shows the behaviour of dynamic system (11). The trajectories near the fixed points can be seen in detail in Figure 2. $P_1$ is a stable node at the origin to which all the near trajectories tend to. $P_2$ is a saddle point on the right branch ($R > 0$) of the null discriminant curve. This point attracts close trajectories towards the curve and deviates them to the origin or to infinity.

Therefore, using model (6) the velocity gradient tensor evolves towards a configuration with two equal positive eigenvalues and one negative in the real case. This implies a saddle/saddle/unstable node topology corresponding to a flow expanding in two spatial directions and contracting in the third.

In the complex case, $R > 0$ produces a configuration with one negative real eigenvalue and two complex with positive real part which corresponds to a topology of unstable focus/contracting. In this case this means rotation with increasing radius in a plane and contraction along the third axis.

Figure 3 shows a scatter plot of invariants $Q$ and $R$ at the final time of their evolution using model (11) from an initial Gaussian velocity gradient distribution. This result is similar to that of Soria *et al* (1994) found in mixing layer simulations.

## CLOSING REMARKS

The rate-of-strain, i.e. the symmetric part of the velocity gradient tensor, has always three real eigenvalues which, for the incompressibility, sum zero. Thus the largest is positive, the smallest negative and the intermediate can be positive or negative. The above description of the resulting topology for the velocity gradient with the proposed model implies for the strain rate more probability of having two positive eigenvalues and one negative than the other possibility (two negative and one positive). This is a well known feature of homogeneous turbulent field in numerical and experimental published results.

The equations obtained by Cantwell (1992) for the invariant evolution in the inviscid case can be derived from the present model by neglecting $\omega_0$. The result is

$$\dot{Q} = -3R$$

$$\dot{R} = \frac{2}{3}Q^2 \qquad (13)$$

With this last system invariants move in phase space along constant discriminant trajectories. The only fixed point in Cantwell's result is a degenerate saddle in the origin and all trajectories go to infinity along the right branch of null discriminant curve. This behaviour, shown in figure 4 is similar to the eq. (11) for high and moderate amplitudes, while for smaller initial values of the invariants the model (11) is convergent, due to the stable node in the origin.

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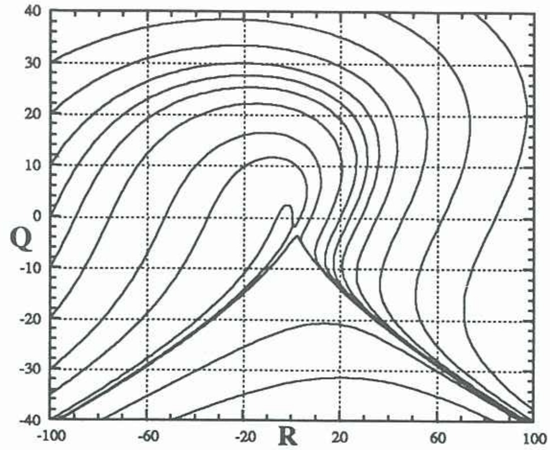


Fig. 1. R-Q Trajectories in phase space using the LMSE model for the diffusion of velocity gradients

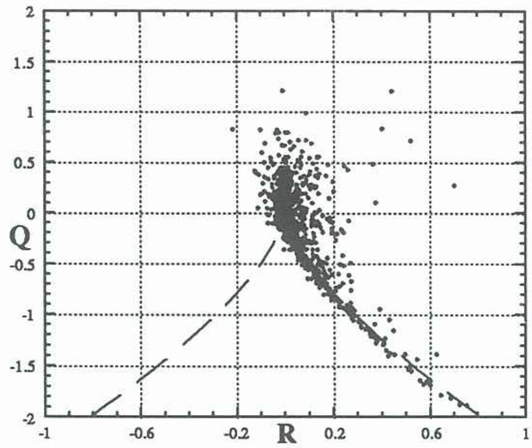


Fig. 3. Scatter plot of R-Q system at the final time of the evolution using the LMSE model from an initially Gaussian velocity gradients distribution.

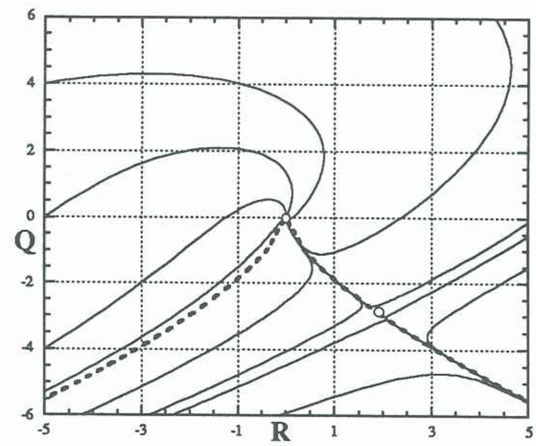


Fig. 2. R-Q Trajectories from LMSE formulation (solid line) near the critical points (circles), and D=0 curve (dashed line)

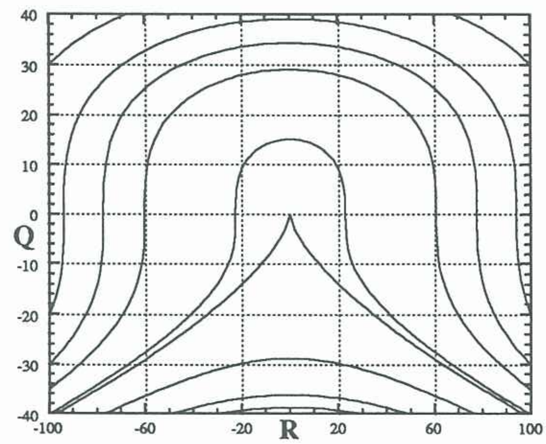


Fig. 4. Trajectories in R-Q phase space with the system of Cantwell (1992) for the inviscid case.