Twelfth Australasian Fluid Mechanics Conference
The University of Sydney, Australia 1995

NUMERICAL SOLUTION OF CONTINUOUS MODEL FOR FLUID–PARTICLE FLOWS

F. Lalli

Istituto Nazionale per Studi ed Esperienze
di Architettura Navale

Roma

Italy

ABSTRACT

In the present paper a numerical solution for continuous model describing the motion of liquids with large amounts of suspended solids is presented. The mathematical formulation is based on Bingham visco–plastic model. The analisys has been carried out by direct simulation of 2-D equations, by means of a finite difference method [9]. Numerical results have been obtained for Bingham flow in a driven cavity and for the interaction of a free surface water wave with sandy bottom.

INTRODUCTION

The fluid mechanical theory of the dynamical interaction of a collection of particles and the surrounding fluid has been developed by several authors [4], by means of a representation of the system as two interpenetrating and interacting continua. The equations of continuity and motion for the model stated in these terms are, as might be expected, considerably more complicated with respect to the case of a single fluid. In particular it is necessary to obtain constitutive relations for the stress tensors of both phases, as well as for the interaction force, and to solve the system of equations (in 3–D, 8 scalar equations) with suitable initial and boundary conditions.

In several practical applications the details of the flow and the motion of particles are not required, whereas what is desired to know is the collective particles motion; a single fluid model, in which the effects of the particulate phase is accounted for in terms of an effective viscosity, can be a rather effective tool. The mechanics of a collection of particles interacting with the surrounding fluid can be therefore studied regarding the two phases as jointly forming a single pseudo–continuum. The literature contains many discussions about this approach, the majority concerned with the relation between the rheology of suspensions and the properties of the fluid and the particles; among these, one could mention the classic paper by Einstein [1] concerning the viscosity of dilute suspensions, and a more recent work due to Happel and Brenner [8], in which the results obtained by Einstein are extended for higher concentrations.

In this paper the numerical solution of a model based on such kind of approach is presented. In particular, a fluid with rheological properties connected to density has been considered: in the regions of dilute suspensions the fluid is assumed to be Newtonian, with viscosity related to concentration [8], whereas, for very high amount of suspended particles, the intergranular forces are represented by Bingham model.

In the first numerical example the simulation of constant density Bingham fluid in a driven cavity has been performed, to verify the capability of the algorithm to deal with strong viscosity variations flows; this test gives rather encouraging results, in that the generation of regions of rigid motion inside the fluid domain does not give rise to numerical instabilities. Finally, in the second example the interaction of gravity wave with bottom particles has been considered.

MATHEMATICAL FORMULATION

Fluid–particle interaction is studied by modeling the two–phase flow by means of a pseudo–continuous medium; the mathematical model can be outlined as follows:

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{0.1}$$

$$\frac{\partial \rho}{\partial t} + (u_i + \tilde{w}_i)\frac{\partial \rho}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\epsilon \frac{\partial \rho}{\partial x_i}\right) \tag{0.2}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial}{\partial x_j}(-p\delta_{ij} + T_{ij}) \tag{0.3}$$

where $u_i = (u, v)$ is the fluid velocity, ρ is the density, p is the pressure (including gravity potential), $w_i = (0, \tilde{w})$ is the settling velocity, ϵ is the particle diffusivity and T_{ij} is the extra–stress tensor.

The main feature of grains with respect to fluids is that frictional shear stress can act even at rest (Coulomb law, [6]); some mathematical aspects of continuum models for the motion of non–cohesive granular materials have been outlined by Jackson [6]: the extra–stress tensor can be written as:

$$T_{ij}: \quad p\sin\phi \begin{pmatrix} -\cos 2\gamma & \sin 2\gamma \\ \sin 2\gamma & \cos 2\gamma \end{pmatrix} \qquad (0.4)$$

where ϕ is the angle of internal friction and γ is the orientation of the principal stress axes. By this constitutive relation the equations of motion can be obtained:

$$\rho\left(\frac{\partial u}{\partial t}+u_j\frac{\partial u}{\partial x_j}\right)=-\frac{\partial p}{\partial x}-\sin\phi\left(\cos 2\gamma\frac{\partial p}{\partial x}-\right.$$

$$\left.\sin 2\gamma\frac{\partial p}{\partial y}-2p\sin 2\gamma\frac{\partial \gamma}{\partial x}-2p\cos 2\gamma\frac{\partial \gamma}{\partial y}\right) \qquad (0.5)$$

$$\rho\left(\frac{\partial v}{\partial t}+u_j\frac{\partial v}{\partial x_j}\right)=-\frac{\partial p}{\partial y}-\sin\phi\left(-\sin 2\gamma\frac{\partial p}{\partial x}-\right.$$

$$\left.\cos 2\gamma\frac{\partial p}{\partial y}-2p\cos 2\gamma\frac{\partial \gamma}{\partial x}+2p\sin 2\gamma\frac{\partial \gamma}{\partial y}\right) \qquad (0.6)$$

These equations, together with [0.1] and the condition of coaxiality:

$$\cos 2\gamma\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\sin 2\gamma\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right) \qquad (0.7)$$

describe the motion of a non–cohesive granular material (coarse sand, $\rho \simeq$ const.).

Numerical strategies to solve these equations, as far as the author knows, are not so well established; therefore in the present work the dynamical behavior of liquids with large amount of suspended solids has been studied by means of Bingham model [2], suitable to represent a large class of fluids, for which some theoretical [3] as well as numerical [5] work has been done. This model is characterized by the presence of a yield stress, which can represent the typical mechanical properties of granular materials, as internal friction and cohesion.

In the expression chosen for the extra–stress tensor T_{ij} viscosity variations with density are taken into account [8] and, beyond a certain threshold of concentration, the viscoplastic behavior does appear. Namely, the incompressible generalized Newtonian fluid constitutive equation is assumed [7]:

$$T_{ij}=2\mu S_{ij} \qquad S_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \qquad (0.8)$$

where the viscosity is a function of density and of the second invariant $S_{II}=\frac{1}{2}S_{ij}S_{ij}$:

$$\begin{cases} \mu=\mu_0(\rho)+\frac{1}{2}\tau_0(\rho)S_{II}^{-\frac{1}{2}} & \text{if } T_{II}\geq\tau_0^2 \\ Sij=0 & \text{if } T_{II}<\tau_0^2 \end{cases} \qquad (0.9)$$

It could be noted that in [0.9] μ is not defined for $S_{II}=0$ (rest or solid–body motion); therefore, such expression has been suitably regularized:

$$\mu=\mu_0(\rho)+\frac{1}{2}\tau_0(\rho)(S_{II}+\eta)^{-\frac{1}{2}} \qquad (0.10)$$

where η can be as small as one wishes.

NUMERICAL EXAMPLES

Primitive equations for the 2–D problem have been discretized on a staggered grid by a finite difference scheme, following Rai and Moin [9].

The first numerical example concerns the case of Bingham flow in a driven cavity (ρ = const.); the solution has been computed in a cartesian uniform 40×40 grid. In fig.1 the streamlines of the steady flow are depicted: the well known flow features of the $Re = 1$. Newtonian case (a) are compared with the viscoplastic case (b: $Re = 1.$, $\tau_0 = 7.5$); the thick line is the boundary of solid–body motion regions. Similar results have been obtained in [5]. In fig.2 the same comparison is pointed out for the horizontal component of velocity in the vertical midsection, whereas in fig. 3 the influence of the parameter η on velocity profile is stressed. The behavior seems to be rather good, since the solution converges rather rapidly (with respect to the parameter η) and it is not necessary to use very small values of such parameter. In the present work an explicit algorithm has been used, and of course the convergence of the solution to the steady state is rather slow, since small time steps have to be used: numerical experiments suggest the use of $\Delta t \simeq .20986 \cdot \Delta x^{2.32193} \cdot \eta^{\frac{1}{2}}$, for $Re = 1.$ and $\tau_0 = 7.5$. In fig.4 the convergence of the solution is shown for various values of the parameter η.

Finally, the numerical algorithm have been implemented to study the interaction between a gravity wave flow and a sandy bottom, with generation of sand waves (fig.5).

AKNOWLEDGEMENTS

The work was supported by Italian Ministry of Merchant Marine in the frame of INSEAN research plan 1991-93.

References

[1] Einstein A., Ann. Phys., vol. 19, 289 (1906).

[2] Bingham E.C., Fluidity and Plasticity, McGraw–Hill, N.Y. (1922).

[3] Oldroid J.G., Proc. Camb. Phil. Soc., **43** (1947), **44** (1948), **45** (1949), **47** (1951).

[4] Davidson J.F., Harrison D., Fluidization, Academic Press, London, 1971.

[5] Bercovier M., Engelman M., A Finite Element Method for Incompressible Non–Newtonian Flows, *J. of Computational Physics*, 36, 1980.

[6] Jackson R., Some Mathematical and Physical Aspects of Continuum Models for the Motion of Granular Materials, Theory of Dispersed Multiphase Flow, Academic Press, 1983.

[7] Bird R.B., Armstrong R.C., Hassager O., Dynamics of Polymeric Liquids, vol. 1, J. Wiley & Sons, N.Y. 1987.

[8] Happel J., Brenner H, Low Reynolds Number Hydrodynamics, M. Nijhoff Pub., 1986, Boston.

[9] Rai M.M., Moin P., Direct Simulations of Turbulent Flow Using Finite–Difference Schemes, *J. of Computational Physics*, 1991.

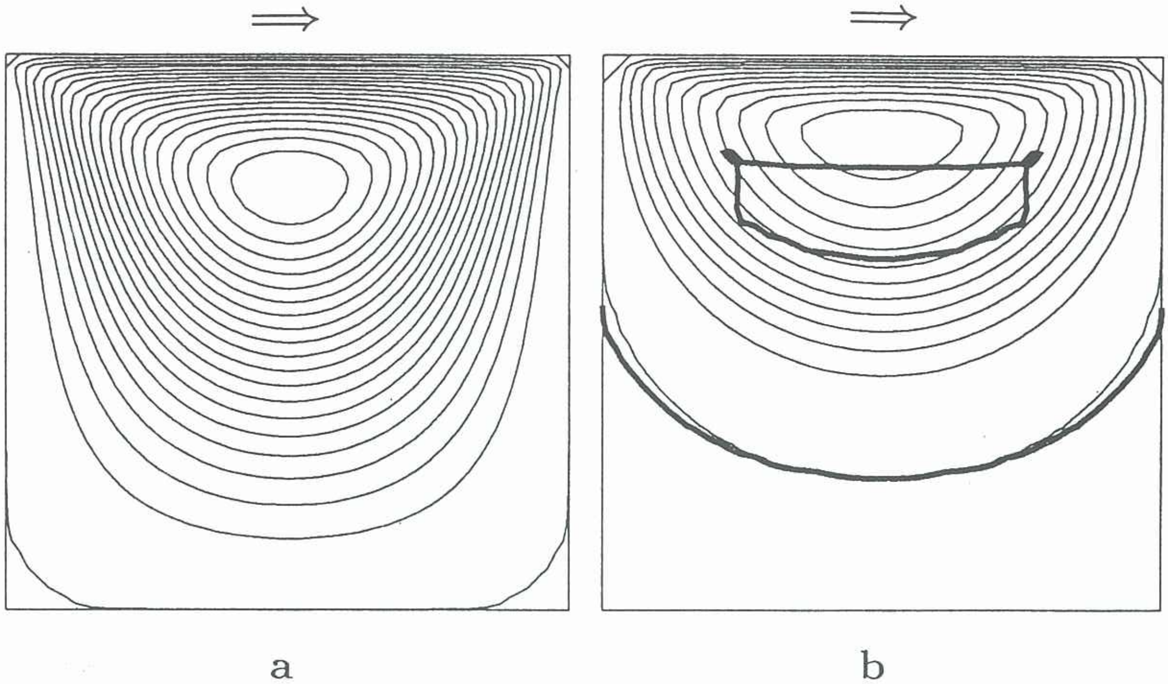

Fig.1 Steady flow streamlines in a driven cavity; (a): Newtonian fluid, $Re = 1.$, (b): Bingham fluid, $Re = 1.$, $\tau_0 = 7.5$.

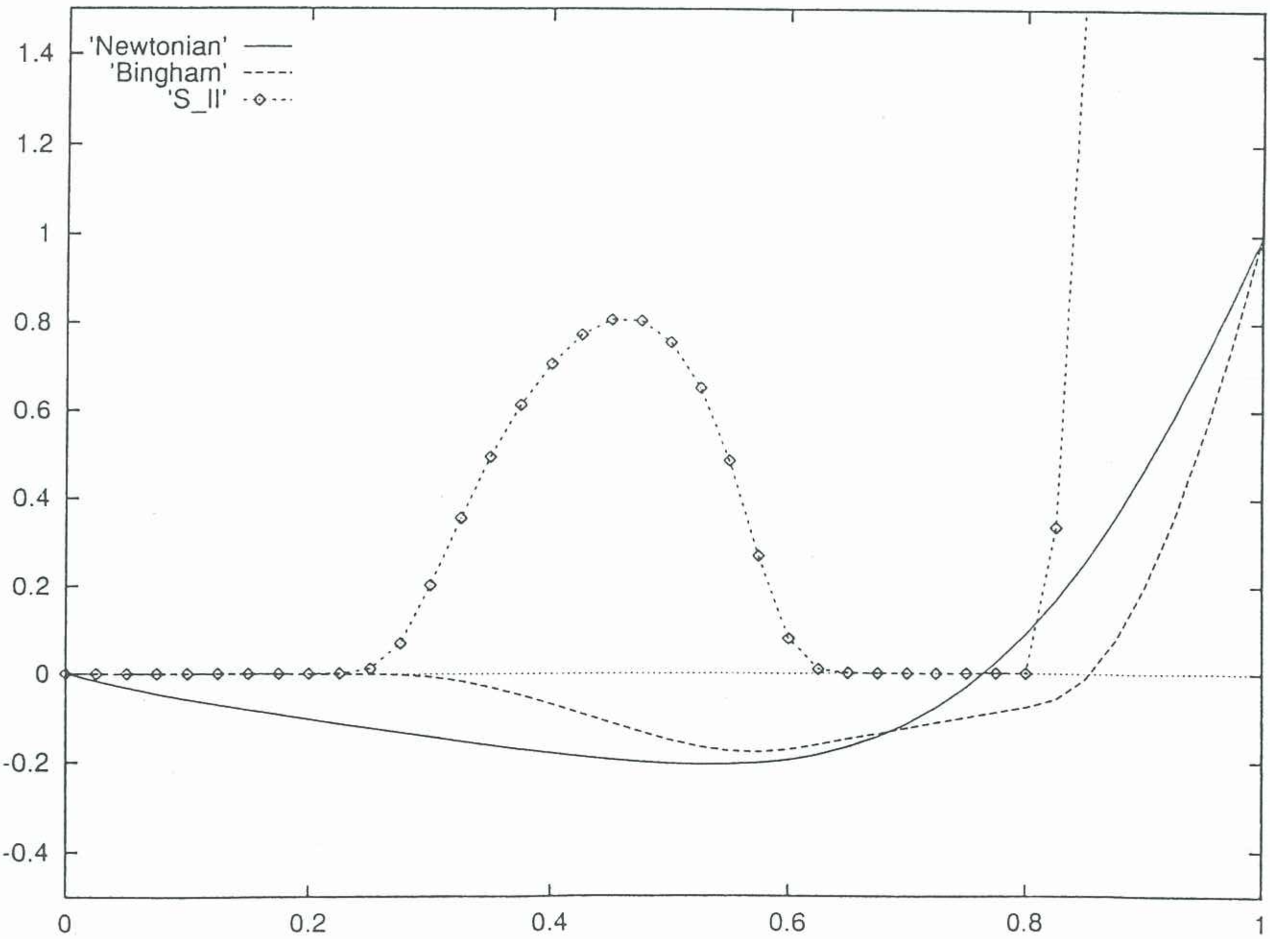

Fig.2 Driven cavity steady flow: profiles in the vertical midsection of S_{II} (Bingham) and horizontal velocity.

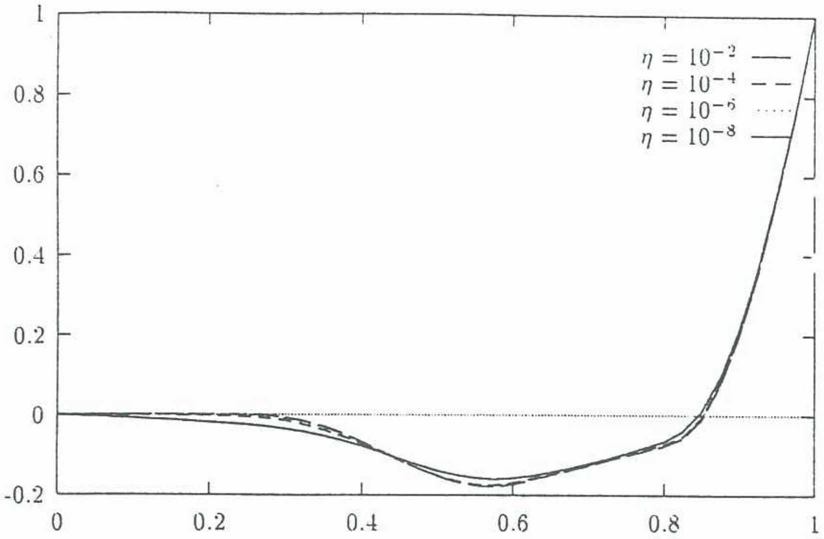

Fig.3 Driven cavity steady flow: horizontal velocity profiles for Bingham flow, $Re = 1.$, $\tau_0 = 7.5$.

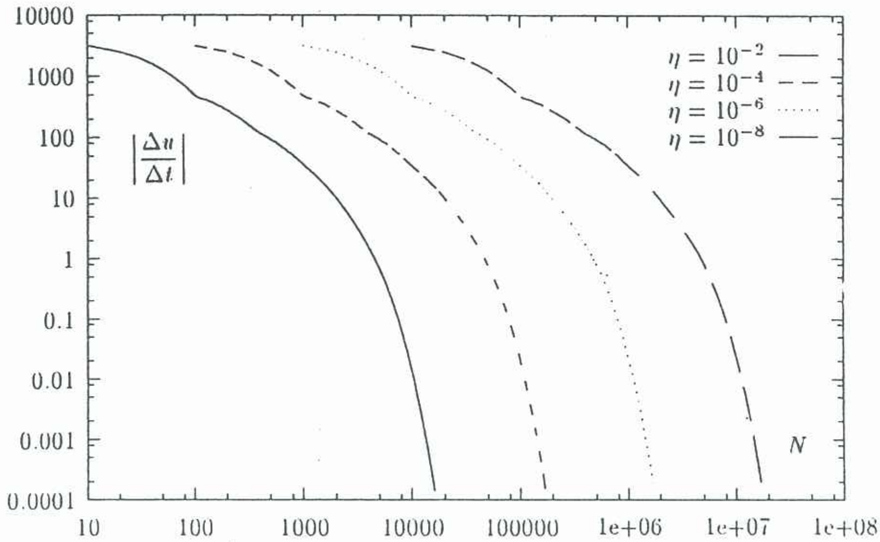

Fig.4 Driven cavity steady flow: convergence to the steady–state solution for Bingham flow.

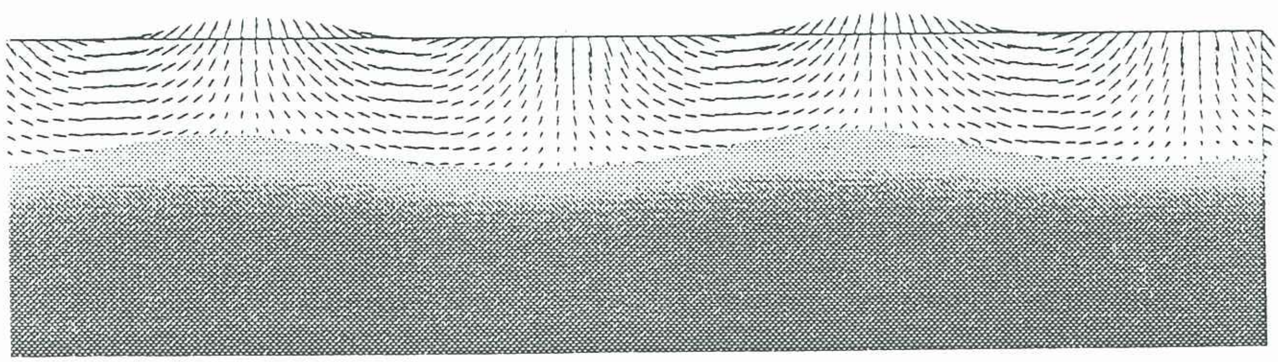

Fig.5 Interaction between gravity wave and sandy bottom; $\Delta\rho_0/\rho_0 = .25$, $\Delta\mu_0/\mu_0 = 10$, $Re = 500$, $Sm = 10$, $\tau_0 = 10$, $\eta = 10^{-4}$.