Twelfth Australasian Fluid Mechanics Conference
The University of Sydney, Australia 1995

VORTICITY FIELD AND IDENTIFICATION OF TWO-DIMENSIONAL VORTICES

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ABSTRACT

We propose and study a new criterion based on the vorticity field properties, in order to define and characterize more precisely vortices (organized structures) in two-dimensional incompressible flows. This criterion is compared with the Weiss criterion. Analytical derivations and numerical studies bring more insights into the viscous vorticity dissipation process.

BASIC EQUATIONS

We consider a two-dimensional incompressible flow; we introduce a stream function $\psi(x,y,t)$ and the vorticity field $\omega(x,y,t)$ such that

$$u(x,y,t) = -\frac{\partial\psi}{\partial y}(x,y,t)$$

$$v(x,y,t) = \frac{\partial\psi}{\partial x}(x,y,t)$$

$$\omega(x,y,t) = \nabla^2\psi(x,y,t)$$

where (x,y) denotes cartesian coordinates, t is the time and (u,v) are the velocity components.

By applying the curl and the divergence operators to the Navier-Stokes equation, we obtain

$$(1) \qquad \frac{\partial\omega}{\partial t} + \frac{\partial\psi}{\partial x}\cdot\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\cdot\frac{\partial\omega}{\partial x} = \nu\,\nabla^2\omega$$

$$(2) \qquad \nabla^2 P = 2\left(\frac{\partial^2\psi}{\partial x^2}\cdot\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x\,\partial y}\right)^2\right)$$

where P(x,y,t) is the pressure field.

The study here proposed is a kinematic one: time is fixed and remains equal to t_0 ; consequently in all the functions involved we no more specify the time variable.

We first recall some elementary results about functions which will be usefull later: if f(x,y) is a given (regular enough) real function, we consider the corresponding Hessian function

$$H(f) = \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2$$

- *First property:* The weak extrema of $\| \nabla f \|^2$ are defined by the condition : $\nabla(\| \nabla f \|^2) = 0$ which results in a linear homogeous system of equations satisfied by the components of ∇f. These weak extrema are different from zero if and only if H(f) is equal to zero.

- *Second property:* The sign of H(f) is the same as the sign of the Gaussian curvature of the surface (S) defined by $z = f(x,y)$ in the three-dimensional space. Consequently the regions defined by $H(f) > 0$ are the only ones which contain closed and convex f-isolines.

- *Third property:* If f satisfies elliptical symmetry properties, i.e. $f(x,y) = g(\rho)$ with $x = a\rho\cos\tau$, $y = b\rho\sin\tau$, then:

$$(3) \qquad \| \nabla f \|^2 = [g'(\rho)]^2\left[\frac{\cos^2\tau}{a^2} + \frac{\sin^2\tau}{b^2}\right]$$

$$(4) \qquad H(f) = \frac{1}{ab\,\rho}\,g'(\rho)\,g''(\rho) = \frac{1}{2ab\,\rho}\frac{d}{d\rho}(g'(\rho))^2$$

STREAM FUNCTION, VELOCITY FIELD AND WEISS CRITERION

Here f is equal to the stream function $\psi(x,y,t_0)$. The Weiss criterion $H(\psi) > 0$ (Weiss, 1981, 1991) has been extensively used in order to identify organized structures (Chong et *al.*, 1990, Jeong and Hussain, 1994). It is based on dynamical arguments and, at least for two-dimensional flows, the validity of these arguments appears to be limited (Basdevant and Philipovitch, 1994). For 2-D flows, it can be recovered in a geometrical way without any restriction: these structures are characterized by closed convex streamline structures owing to the "second property" previously quoted.

It is well known that such structures can be found in low pressure regions, but never exist in high pressure regions. This result can be analytically derived from the relation (2) $\nabla^2 P = 2H(\psi)$ (Larchevêque 1993).

Moreover the points defined by $H(\psi) = 0$ correspond to weak extremum values for the kinetic energy $\|\nabla\psi\|^2$ and appear to be exactly local maxima of $\|\nabla\psi\|^2$ if ψ satisfies elliptical symmetry properties, according to the relations (3) and (4). We may notice that in that case $\|\nabla\psi\|^2$ is equal to zero in the central point of the elliptical structure.

VORTICITY FIELD AND VORTICITY DISSIPATION

Now f is equal to $\omega(x,y,t_0)$. The Hessian function $H(\omega)$ introduces a new criterion for the identification of vortices. This criterion has been proposed by Larchevêque (1993) and brings a topological definition of vortices: such vortices correspond to closed convex isovorticity lines; they are embedded in the flow regions characterized by $H(\omega) > 0$ (second property recalled in § 1). Like the Weiss criterion, this criterion satisfies Galilean invariance and is more closely related to the small scale details of the flow than the Weiss criterion is.

By exchanging the stream function into the vorticity function, the same kind of analytical derivations show that all the points of the flow defined by the condition $H(\omega) = 0$ are associated with weak extremum values for $\|\nabla\omega\|$. When ω exhibits elliptical symmetry properties, the points corresponding to the condition $H(\omega) = 0$ define local maximum values of $\|\nabla\omega\|$. We can conclude that, at least for elliptical vortices, the condition $H(\omega) > 0$ better educes the vortex cores than the Weiss criterion does since, following Dritschel (1993), the vortex edge can be defined by the location of the peaks of the vorticity gradient.

Now we consider the viscous dissipation of the squared vorticity for structures exhibiting an elliptical symmetry. The equation (1) leads to

$$\frac{d}{dt}\omega^2 = 2\nu\,\omega\,\nabla^2\omega = \nu\,\nabla^2\omega^2 - 2\nu\,\|\nabla\omega\|^2 \tag{5}$$

where $\frac{d}{dt}$ denotes the material derivative.

At the central point of such structures, the gradients of $\omega\nabla^2\omega$ and $\nabla^2\omega^2$ are equal to zero as an outcome of symmetry properties. It will appear from numerical studies that $\omega\nabla^2\omega$ and $\nabla^2\omega^2$ take negative values in the vortex cores, with peaks for their absolute values at the vortex central points when $\nabla\omega$ is equal to zero: the local squared vorticity dissipation exhibits a maximum efficiency in the vortex core while the contribution $\|\nabla\omega\|^2$ from this region to the enstrophy dissipation (half mean squared vorticity) is very low.

NUMERICAL STUDIES

We investigate 2-D mixing layer, 2-D Taylor-Green vortex and decaying turbulent flows by performing direct numerical simulations of the two-dimensional Navier-Stokes equation (we do not introduce hyperviscosity). The characteristics of the runs are the following ones:

- First we consider a temporal mixing layer with periodic boundary conditions in the x-direction and slip boundary conditions in the y-direction. The numerical model is a pseudo-spectral one with 512x513 grid points; a third Adams-Bashforth scheme is used for time marching. The initial velocity profile is an error function with two sine pertubations, and the Reynolds number is equal to 200. The final state is a single vortex resulting from the pairing of two vortices (figure 1).
- The 2-D Taylor-vortex run is defined by a 2-D decaying turbulent flow in a 1025x1025 square box with slip boundary conditions. The numerical model remains a pseudo-spectral one with a third Adams-Bashforth time scheme. The initial flow is a 7x7 Taylor-Green vortex stream function which is destabilized by a white noise of small amplitude. Here the Reynolds number is equal to 800. We reache a final state comprising three vortices (figure 2). The two biggest one exhibit a quasi elliptical shape only in their central part.
- In the third run we perform a direct numerical simulation of decaying turbulence in a 1024x1024 square box with periodic conditions. Here again the numerical model is a pseudo-spectral one with a third Adams-Bashforth time scheme. The initial velocity field is a zero-mean Gaussian random function with an energy spectrum $E(k,t=0) = C\,k\,\exp[-(k/k_0)^2]$ where $k_0 = 8$ and $C = 0.2$. The Reynolds number is equal to 1000 and the final stage of the flow evolution is a two vortex state (figure 3).

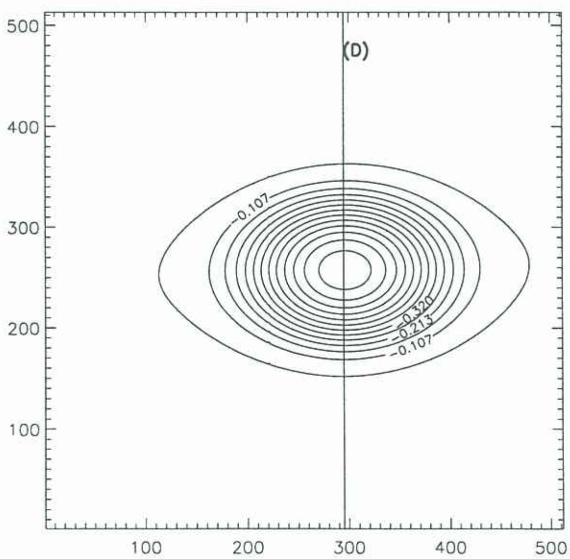

FIGURE 1
2-D MIXING LAYER AFTER VORTEX PAIRING
ISOVORTICITY LINES

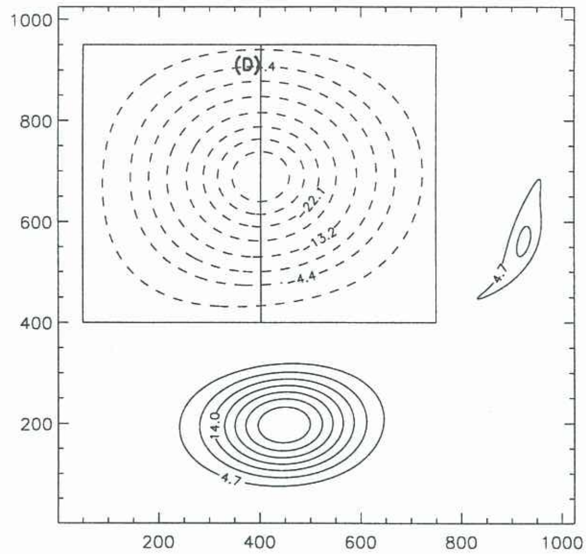

FIGURE 2
2-D TAYLOR-GREEN VORTICES
ISOVORTICITY LINES

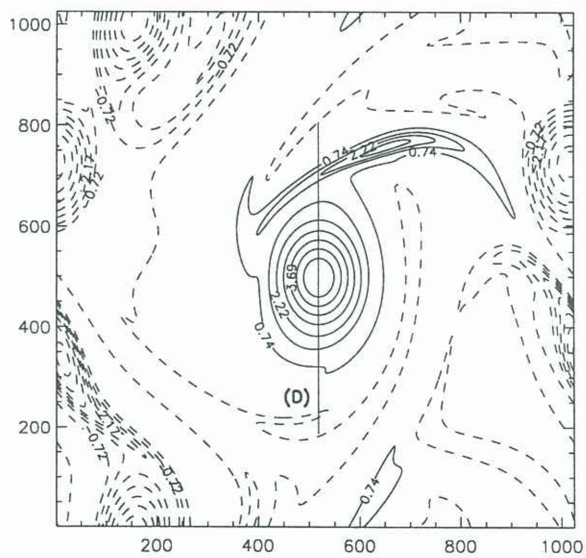

FIGURE 3
DECAYING TURBULENT FLOW
ISOVORTICITY LINES

The numerical results obtained are summarized by the figures 4 to 6 which display the variations of ω, $H(\psi)$, $\|\nabla\psi\|^2$, $H(\omega)$, $\omega\nabla^2\omega$, $-\|\nabla\omega\|^2$ and $\nabla^2\omega^2/2$ in the flows along a straight line D which is drawn on the figures 1 to 3. These different quantities are normalized by the maximum of their absolute values along D, except for $-\|\nabla\omega\|^2$ and $\nabla^2\omega^2/2$ which are normalized by $\max(|\omega\nabla^2\omega|)$.

The three different runs are in very good agreement with the analytical results peviously developped; in particular $\|\nabla\omega\|^2$ exhibits extremum values when $H(\omega)$ becomes equal to zero, even when the structure is not exactly an elliptical one. The central part of the vortices defined by $H(\omega) > 0$ is narrower than the center part defined by $H(\psi) > 0$. The $H(\omega)$ criterion appears to define more precisely the core of the vortices than the Weiss criterion does.

Moreover for the three runs $\nabla^2\omega^2/2$, which reaches extremun negative values at the central point of the vortices, becomes equal to zero near the edges $H(\omega) = 0$ of these vortices. As $\|\nabla\omega\|$ is equal to zero at the central point of the vortices and exhibits maximum values on the vortex edges, the main part of the contribution of these vortices to the the viscous vorticity dissipation comes from the $\nabla^2\omega^2/2$ term in their central part and from $\|\nabla\omega\|^2$ near their edges. We recall that this last term is the only one which brings a contribution to the flow enstrophy dissipation in homogeneous turbulence.

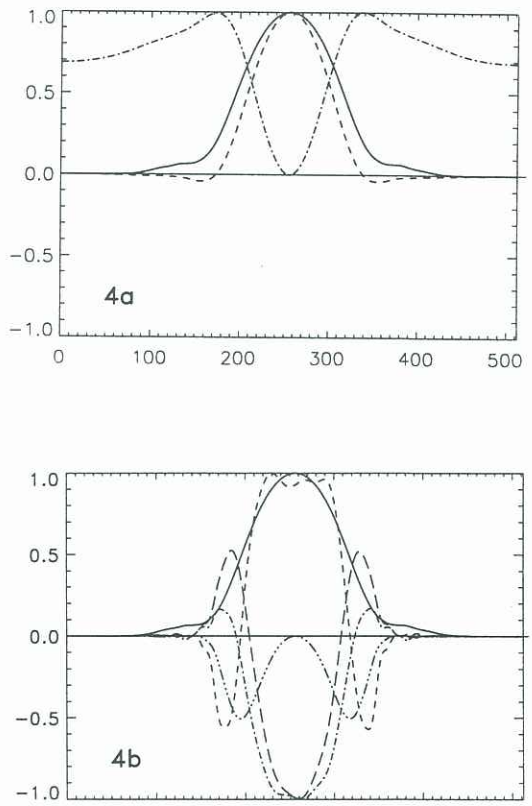

FIGURE 4
2-D MIXING LAYER AFTER VORTEX PAIRING
4 a: —— ω, --- $H(\psi)$, -·-·-· $\|\nabla\psi\|^2$,
4 b: —— ω, --- $H(\omega)$, -·-·-· $\omega\nabla^2\omega$
-·····- $-\|\nabla\omega\|^2$, — — — $\nabla^2\omega^2/2$.

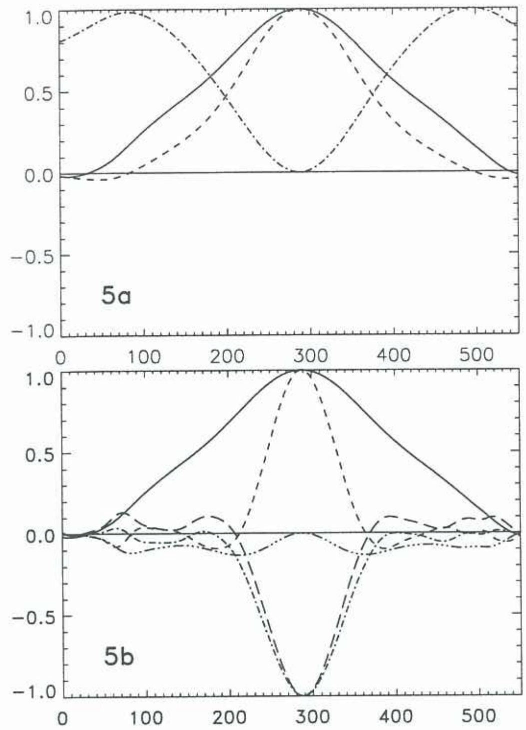

FIGURE 5
2-D TAYLOR-GREEN VORTICES
5 a: —— ω, --- $H(\psi)$, -·-·- $\|\nabla\psi\|^2$,
5 b: —— ω, --- $H(\omega)$, -·-·- $\omega\nabla^2\omega$
-···-··· $-\|\nabla\omega\|^2$, – – – $\nabla^2\omega^2/2$.

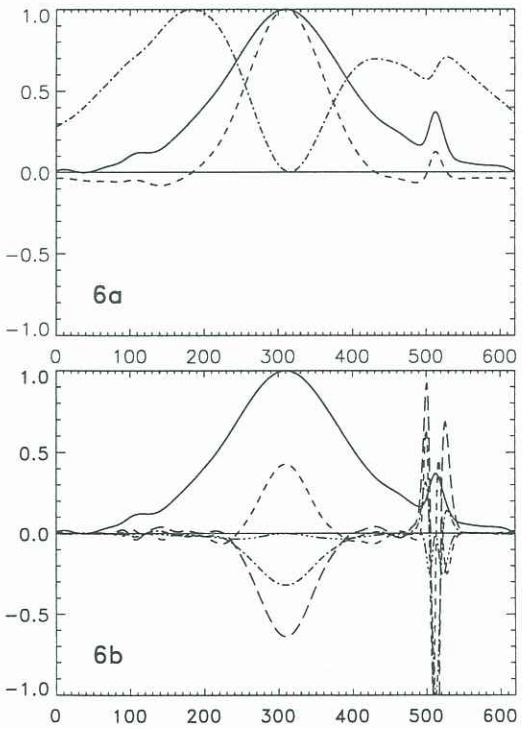

FIGURE 6
DECAYING TURBULENT FLOW
6 a: —— ω, --- $H(\psi)$, -·-·- $\|\nabla\psi\|^2$,
6 b: —— ω, --- $H(\omega)$, -·-·- $\omega\nabla^2\omega$
-···-··· $-\|\nabla\omega\|^2$, – – – $\nabla^2\omega^2/2$.

CONCLUSION

Analytical and numerical derivations show that the new vorticity criterion proposed here defines the vortex edge by peaks of vorticity gradients and better identifies the core of the two-dimensional vortices than the Weiss criterion does. We have also established that the kinetic energy of the flow exhibits peaks at zero pressure Laplacian values when the stream function satisfies elliptical symmetry properties. The different contributions to the viscous dissipation of vorticity in the vortex cores have been also analysed. Nevertheless all these derivations do not bring new information about constant vorticity vortices such as the vortices studied by Kida (1981).

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