MULTIGRID ACCELERATION OF COMPRESSIBLE FLOW COMPUTATION

A. Di Mascio
Istituto Nazionale per Studi ed Esperienze di Architettura Navale
Rome
Italy

B. Favini
Dipartimento di Meccanica ed Aeronautica
Università degli Studi di Roma "La Sapienza"
Rome
Italy

ABSTRACT

In the present work the multigrid strategy is applied to second order ENO schemes to speed up the solution of steady compressible flows. The performances of the algorithm are analysed in several flow situations, ranging from low subsonic regime to high supersonic flow, for both internal and external problems. Computational efficiency and solution accuracy are checked with two different Riemann solvers.

INTRODUCTION

Essentially Non Oscillatory (ENO) schemes (Harten et al., 1987) have become very popular in numerical simulation of inviscid compressible flows during the last decade because of their strong theoretical background, their capability of dealing with complex flows with discontinuities, and their "robustness" without the addition of nonphysical extra terms (artificial dissipation).

All these positive aspects make these algorithms extremely attractive for numerical simulation in unsteady gasdynamics. Conversely, the CPU time requirements is the main shortcoming for steady flow computations. From this point of view, methods based on central differences with artificial dissipation are to be preferred. In fact, besides the low CPU cost per iteration, the use of multigrid algorithms to improve the convergence rate of these schemes is well established. The most popular algorithm of this kind is the Jameson's scheme (Jameson, 1983), based on Runge-Kutta pseudo-time integration, which is, at present, the fastest algorithm in steady compressible flow simulation. More recently, the multigrid algorithm has been applied also to upwind schemes by Dick (1990), who applied the multigrid algorithm to flux-difference splitting methods.

On the ground of the remarkable improvement in performances of centered and upwind schemes with multigrid algorithms, we checked how the multigrid strategy performs when applied to second order ENO schemes. To this end, we studied a Full Approximation Scheme (FAS) with the V cycle.

In the sequel of the paper, we report some numerical tests performed to check the multigrid properties in conjunction with ENO schemes. For the sake of conciseness, neither the numerical scheme nor the multigrid algorithm will be described in details. The reader is referred to (Harten et al., 1987) for a complete analysis of ENO scheme and to (Brandt, 1984) for a detailed discussion on multigrid strategies.

The analysis was limited to second order ENO schemes for two dimensional problems in curvilinear coordinates. The performances of the multigrid algorithm were tested with two Riemann solver, i.e. the exact Riemann solver developed by Gottlieb and Groth (1988), that requires the iterative solution of a nonlinear system at each cell interface, and the solver developed by Harten et al. (1983), that gives an approximate solution in closed form.

NUMERICAL TESTS

The performances of the multigrid algorithm have been tested for various flow regimes, in the whole range from low subsonic to high supersonic Mach number, for both internal and external flows. In all the numerical tests, the initial condition is a uniform flow and convergence is assumed to be obtained when the L_2 norm of the mass convervation residual is reduced by six order of magnitude.

The global efficiency of a multigrid algorithm will be measured in term of the Work Reduction Factor (*wrf* in the following), defined as the ratio of the work needed to reach the steady state in a standard single grid calculation to the work required by a multigrid calculation, where the work unit is defined as the cost of one iteration on the finest grid.

In the test cases reported, the number of levels and the number of iterations at each level will be indicated as $V_{\nu_1/\nu_2/\ldots/\nu_N}$, that means that N levels were used, the number of iteration was ν_1 on the first (the finest) grid, ν_2 on the second, ..., ν_N on the last (the coarsest). For each test case, only the best V cycles will be reported. It is to be noticed that in no cases the V cycle includes smoothing iterations in the ascending phase, i.e. when interpolating the correction from a coarse grid to a finer one. Numerical experience showed that this is the best choice from the point of view of global efficiency.

Finally, the aforementioned Riemann solvers will be referred to as the "exact" solver (Gottlieb, 1988), and the "HLL" solver (Harten et al., 1983).

Internal Flows

Subsonic flow in a channel. The first test was a subsonic flow in a channel with a circular arc bump on the lower wall. The width of the channel is equal to the length of the bump, and the thickness-to-chord ratio of the bump is 10%. The inflow Mach number is 0.5.

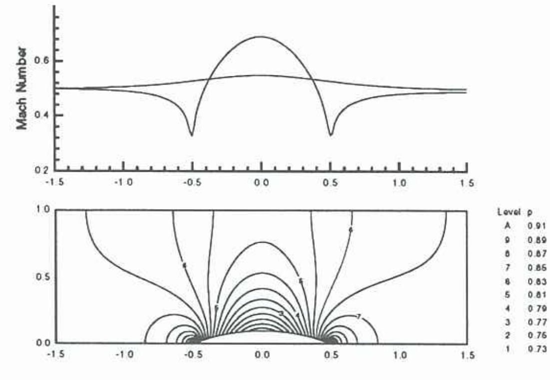

Figure 1: Subsonic flow in a channel. Mach number distribution on upper and lower walls (top half) and pressure contours (bottom half). Exact solver, grid 192×64, $M_\infty = .5$

Grid 192×64

Solver	Cycle	Work	wrf
Exact	SG	76459.00	–
Exact	$V_{5/15/25/50}$	2124.13	36.00
HLL	SG	77335.00	–
HLL	$V_{5/15/25/100}$	633.19	122.14

Grid 96×32

Solver	Cycle	Work	wrf
Exact	SG	25672.00	–
Exact	$V_{5/15/25}$	1098.70	23.36
HLL	SG	24733.00	–
HLL	$V_{5/25/50}$	527.48	46.89

Table I: Efficiency of the multigrid calculation for the subsonic flow in a channel

Figure 1 shows the pressure contours and the Mach number distribution on the lower and upper walls for the converged solution . The same flow was also computed with a 96×32 grid and both the computation were repeated with the HLL solver. In table I the work required by the computations is reported. As it can be inferred from the table, the efficiency considerably grows with the grid size when using the HLL solver, while the increase is less significant with the exact solver. Moreover the performances increase with the number of the grid levels; the maximum was found with the four level computation, in which case a very high work reduction factor is gained ($\sim$ 122 with the HLL solver and $\sim$ 36 using the exact solver). We have also checked the multigrid algorithm with five and six grid levels with the grid 192×64, but it seems that the performances cannot be improved.

The better behaviour of the multigrid strategy in conjunction with the HLL solver is probably to be related to its dissipative properties, which are greater than that of the exact solver, as the lower accuracy in the solution reveals.

Transonic flow in a channel. The same channel geometry was considered for the analysis of a transonic flow computation with inflow Mach number equal to 0.675. Like in the subsonic case, two grids and two Riemann solvers were used to check the performances of the multigrid algorithm.

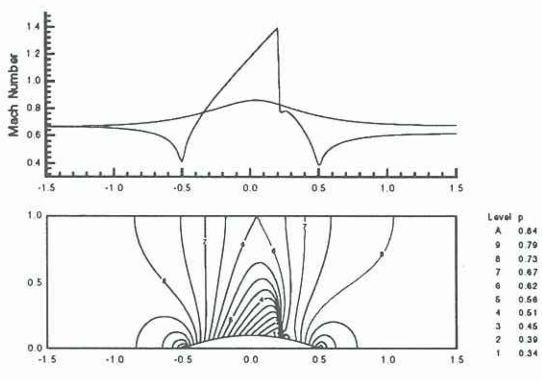

Figure 2: Transonic flow in a channel. Mach number distribution on upper and lower walls (top half) and pressure contours (bottom half). Exact solver, grid 192×64, $M_\infty = .675$.

Grid 192×64

Solver	Cycle	Work	wrf
Exact	SG	34017.00	–
Exact	$V_{5/15/25/50}$	2116.61	16.07
HLL	SG	33800.00	–
HLL	$V_{5/15/25/50}$	1272.64	26.56

Grid 96×32

Solver	Cycle	Work	wrf
Exact	SG	16310.00	–
Exact	$V_{5/25/50}$	1958.40	8.66
HLL	SG	15394.00	–
HLL	$V_{5/10/15}$	1182.98	13.01

Table II: Efficiency of the multigrid calculation for the transonic flow in a channel. Grid 96×32 and 192×64

In figure 2 pressure contours and Mach number distribution on upper and lower walls of the channel are presented.

Multigrid efficiency was tested with the same grids as in the previous case; table II shows the performances that have been obtained. It can be noticed that the reduction factors in the transonic case are smaller than in the subsonic case. Anyhow, at least 90% of CPU time is saved with both solvers in all cases. Besides, the improved performances of the multigrid algorithm when refining the grid and when using the HLL solver are confirmed, and also in this case the improvement with the exact solver is much smaller.

Finally, from table I and table II, it can be noted a reduction in the number of iterations on the coarse levels in the best V-cycle. This is a general trend observed when the Mach number increases.

Supersonic flow in a channel. Figure 3 presents the supersonic flow in a channel with $M_\infty = 1.4$ at inflow. In this case the thickness-to-chord ratio is 4%. The outflow being supersonic, boundary conditions are enforced only at the inflow section ($M_\infty = 1.4$, $\rho_\infty = 1.4$, $p_\infty = 1$).

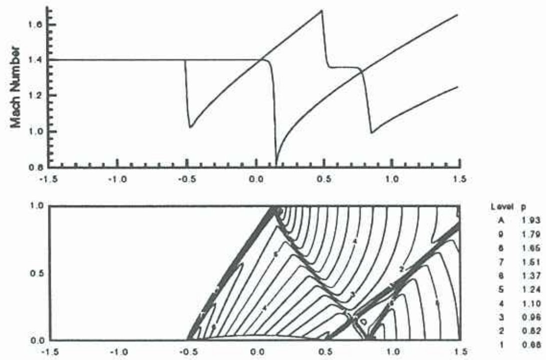

Figure 3: Supersonic flow in a channel. Mach number distribution on upper and lower walls (top half) and pressure contours (bottom half). Exact solver, grid 192×64, $M_\infty = 1.4$.

As to multigrid efficiency, work units for both single grid and multiple grid technique are reported in table III. In the current case, the time reduction factor is much smaller than in the previous tests. It must be noticed that, differently from the previous cases, the reduction factor when using the HLL solver is almost the same as the computation with the exact solver on the grid 96×32, while the HLL calculation is again more efficient on the finer grid 192×64.

Grid 192×64

Solver	Cycle	Work	wrf
Exact	SG	6856.00	–
Exact	$V_{5/10/15/20}$	2288.25	3.00
HLL	SG	6867.00	–
HLL	$V_{5/10/15/20}$	1273.49	5.39

Grid 96×32

Solver	Cycle	Work	wrf
Exact	SG	3444.00	–
Exact	$V_{5/10/15}$	1216.85	2.83
HLL	SG	3378.00	–
HLL	$V_{5/10/15}$	1085.16	3.11

Table III: Efficiency of the multigrid calculation for the supersonic flow in a channel. Grid 96×32 and 192×64

It is to be noted that the number of iterations on the coarse meshes is smaller than in the subsonic and the transonic cases.

External Flows

Transonic flow around a NACA 0012 airfoil. The transonic flow past a NACA 0012 airfoil at $M_\infty = 0.85$ and one degree angle of attack (Dervieux et al., 1989) is a severe test case for inviscid flow solvers because the numerical solution is extremely sensitive to calculation parameters. In the computation reported, the physical domain was discretized by means of an O-type mesh whose external boundary is placed 100 chords far from the airfoil.

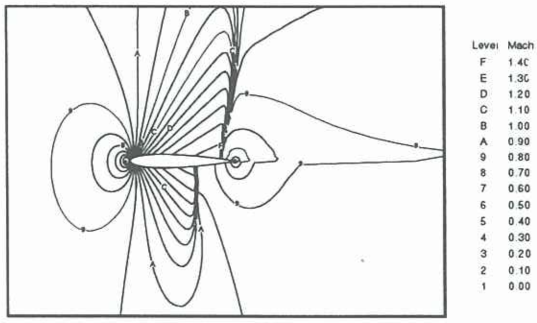

Figure 4: Transonic flow around NACA–0012 airfoil: Mach number contours. Exact solver, $M_\infty = 0.85$, $\alpha = 1°$, O-grid 192×64

Solver	Cycle	Work	wrf
Exact	SG	11378.00	–
Exact	$V_{5/15/25/50}$	1397.38	8.14
HLL	SG	22239.00	–
HLL	$V_{5/15/25/100}$	1331.95	16.70

Table IV: Transonic flow around a NACA–0012 airfoil: multigrid performances

Figure 4 shows the Mach number contours, obtained with the exact solver. The computing times for this test case are shown in table IV. It is interesting to note that, in the single grid calculation, the HLL solver, which is the cheapest of the two for the solution of the single Riemann problem, yields the most expensive global solution. This is due to the reduced stability limit observed when using the HLL solver and to the increased number of iteration required to obtain steady state (compare the work for the two single grid computation in table IV).

As in the previous cases, the reduction factors obtained with the HLL solver are greater than those obtained with the exact solver; however, due to the larger CPU time required in the standard single grid procedure by the HLL solver, the differences in the final cost with the two solvers are practically negligible.

Supersonic flow around a cylinder. Two test-cases with upstream Mach number $M_\infty = 4$ and $M_\infty = 10$ are presented. The physical domain is discretized with a C type grid with 192×128 cells.

Undisturbed flow is assumed at the parabolic outer boundary, while the variables are extrapolated on the two vertical lines (supersonic outflow conditions); zero normal flow is enforced on the solid wall of the cylinder.

Only the HLL solver calculation is reported, because we met with difficulties when using the exact solver. In fact, while we always obtained a converged solution with the HLL

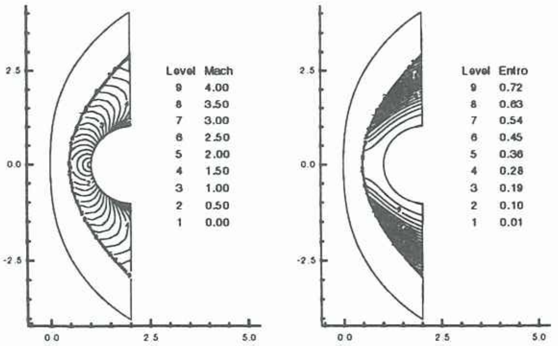

Figure 5: Supersonic flow around a cylinder: Mach number (right) and entropy (left) contours. HLL solver, $M_\infty = 4$, C-grid 192×128.

solver, the computations with the other solver often failed to reach the steady state, without any sistematic dependence of convergence on the grid size and shape, or on the free stream Mach number. However, such troubles are not surprising with this type of flow. Quirk (1992), in his "catalogue of failings", reports the so-called "carbuncle phenomenon" as an example of computational failing of Godunov-type schemes.

Figure 5 shows Mach number and entropy contours in the case $M_\infty = 4$. The flow with $M_\infty = 10$ has the same structure as in the case $M_\infty = 4$, except that the bow shock is closer to the cylinder.

In table V the CPU cost and the work units for the current test are reported. Like in the test case of the supersonic flow in the channel, the higher the upstream Mach number, the lower the multigrid efficiency. We can observe that the loss of efficency is dramatic: in fact, the work at $M_\infty = 10$ is three times larger than in the case $M_\infty = 4$. It must also be remarked that in the best cycle at $M_\infty = 10$ the number of iterations decreases from the finest to the coarsest level (table V), i.e. the structure of the optimum V cycle is reversed with respect to lower Mach number computations.

$M_\infty = 4$

Cycle	Work	wrf
SG	12644.00	–
$V_{5/15/25}$	1298.15	9.74
$V_{5/10/15/25}$	999.52	12.65
$V_{5/10/15/25/30}$	967.40	13.07

$M_\infty = 10$

Cycle	Work	wrf
SG	11524.00	–
$V_{15/10/5}$	3867.11	2.98
$V_{30/15/10/5}$	3255.37	3.54
$V_{30/20/15/10/5}$	2527.19	4.56

Table V: Supersonic flow around a cylinder. Multigrid efficiency with the HLL solver

CONCLUSIONS

In the present work we have analysed the behaviour of the multigrid strategy when applied to second order ENO-type schemes. As expected, the efficiency of the multigrid algorithm was extremely high when dealing with fully subsonic problems, for which the steady state Euler equations are elliptic everywhere in the field.

The algorithm works equally well, although with reduced efficiency, when dealing with transonic flows, i.e. with mixed type problems. Numerical experiments reveal that a computation ten time as cheap as the single grid computation is to be expected in most cases.

The efficiency of the multigrid algorithm decreases further when calculating external supersonic flows; however, a ten times reduction in CPU time is still obtained if the Mach number is smaller than $3 \sim 4$. Conversely, it was found that the performances are much worse when calculating external supersonic flows past blunt bodies with Mach number greater than $8 \sim 10$ or internal supersonic flow past slender obstacles. In these cases, the reduction factor to be expected is at most 5. The lowering of multigrid effectiveness is to be related to the contraction of the subsonic region in external supersonic flows or to its almost total absence for the case of internal flows past slender obstacles.

It is interesting to remark that the structure of the "optimal" V cycle regularly changes from the usual cycle with few iterations on the fine grid and many more smoothing steps on the coarse levels in subsonic flow regimes, to a structure which is completely reversed in high supersonic flows.

Regarding the performances of the multigrid algorithm with different Riemann solvers, the best efficiency was always obtained with the HLL solver, which is also the less expensive for the solution of the single Riemann problem. However, this favourable aspect is often counterbalanced by a reduced accuracy in the numerical solution.

AKNOWLEDGEMENTS

This work was supported by the Italian Ministry of Transportation and Navigation in the frame of INSEAN research plan 1991-93.

REFERENCES

Brandt A., 1984, "Multigrid Techniques: 1984 Guide with Application to Fluid Dynamics", The Weizmann Institute of Science, Rehovot (Israel).

Dervieux A., Van Leer B., Periaux J., Rizzi A. Eds., 1989, "Numerical Simulation of Compressible Euler Flows", *Notes on Numerical Fluid Mechanics*, Vol.26, Vieweg.

Dick E., 1990, "Multigrid Formulation of Polynomial Flux–Difference Splitting for the Steady Euler Equations" *J. Comp. Phys.*, 91, 161–173.

Gottlieb J.J., Groth C.P.T., 1988, "Assessment of Riemann Solvers For Unsteady One–Dimensional, Inviscid Flows of Perfect Gases", *J. Comp. Phys.*, **78**, 437.

Harten A., Lax P.D. and Van-Leer B., 1983, "On Upstream Differencing and Godunov-Type Schemes for Hyperbolics Conservation Laws", *SIAM Rev.*, 25, 35-61.

Harten A., Engquist B., Osher S., Chakravarthy S.R.,1987, "Uniformly High Order Accurate Essentially Non–Oscillatory Schemes", *J. Comp. Phys.*, 71, 231.

Jameson A., 1983, "Solution of the Euler Equation for Two Dimensional Transonic Flow by a Multigrid Method" MAE Report 1631, Princeton University, Princeton.

Quirk J.J., 1992, "A Contribution to the Great Riemann Solver Debate", ICASE Report No. 92-64.