NUMERICAL SIMULATION OF A

SPATIALLY-EVOLVING BOUNDARY LAYER

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ABSTRACT

A numerical method for the simulation of a spatially evolving boundary layer is presented. The governing equations are cast in velocity-vorticity form to avoid complications associated with calculating the pressure. The numerical method uses a mixed spectral/compact difference scheme for evaluating the spatial derivatives. Compact differences are used in the streamwise direction to capture the spatial evolution of the flow. Jacobi and Fourier spectral methods are used in the wall-normal and spanwise directions respectively. At the outflow boundary, a non-reflective boundary condition is imposed to prevent reflection of outgoing waves and their unstable interaction with the flow inside the solution domain. Time-stepping is fully explicit. The numerical method is tested by solving both the Orr-Sommerfeld equation and the full Navier-Stokes equations for steady-flow solutions. Agreement between computations and theory and other published results is good.

INTRODUCTION

To investigate the properties of coherent structures in transitioning and turbulent boundary layers, in particular their soliton-like properties (Kachanov 1994, Bulbeck et al. 1994) and other issues related to turbulent fine-scale motions, an accurate numerical method has been developed to simulate a spatially-evolving incompressible boundary layer. The accuracy of the numerical method derives from the spectral discretisation of the spatial derivatives in the wall-normal and spanwise directions, and high-order compact difference schemes in the streamwise direction. For accurate direct numerical simulation of boundary layer flows, researchers have in the past used spectral methods in all three directions (e.g. Spalart et al. (1991), Laurien and Kleiser (1989)). However, the use of a Fourier spectral method in the streamwise direction assumes periodicity in this direction, and only temporally-evolving boundary layers can be simulated. Recently, Joslin et al. (1992) developed a mixed spectral/compact-difference method (using Chebyshev polynomials with an algebraic mapping in the wall normal direction) in which the boundary layer was allowed to evolve in the streamwise direction. They solved the equations in primitive-variable form using a time-splitting scheme and used a buffer-domain technique to avoid reflections of outgoing waves.

The present work adapts the method of Buell (1994) to spatially-evolving boundary layer flows. High accuracy is achieved by using expansion function based on Jacobi polynomials with an exponential mapping which clusters grid points close to the wall (Spalart et al. 1991). Time-splitting errors and complications associated with solving for pressure are avoided as the equations are cast in velocity-vorticity form. The problem of outgoing waves is treated by using a non-reflective outflow boundary condition, thus avoiding the need for a buffer-domain and the associated memory and CPU requirements.

GOVERNING EQUATIONS

The simulated flow is three-dimensional and unsteady and so the governing equations are the full incompressible Navier-Stokes equations. The equations to be advanced in time are derived by taking the curl of the momentum equations twice and retaining the wall-normal components of velocity and

vorticity. They are:

$$\frac{\partial \nabla^2 v}{\partial t} = \frac{\partial^2 H_1}{\partial x \partial y} - \frac{\partial^2 H_2}{\partial x^2} -$$

$$\frac{\partial^2 H_2}{\partial z^2} + \frac{\partial^2 H_3}{\partial y \partial z} + \frac{1}{Re} \nabla^4 v \qquad (1)$$

and

$$\frac{\partial \omega_y}{\partial t} = \frac{\partial H_3}{\partial x} - \frac{\partial H_1}{\partial z} + \frac{1}{Re} \nabla^2 \omega_y \qquad (2)$$

with boundary conditions

$$u, v, w = 0 \text{ at } y = 0 \text{ and}$$
$$u \to U_\infty \text{ and } v, w \to 0 \text{ as } y \to \infty.$$

x is the streamwise co-ordinate, y the wall-normal co-ordinate and z the spanwise co-ordinate. The fluid velocity is $\bar{u} = \{u, v, w\}$ and the fluid vorticity is $\bar{\omega} = \{\omega_x, \omega_y, \omega_z\} = \nabla \times \bar{u}$. $\bar{H}$ is the vector of the nonlinear terms, $\bar{\omega} \times \bar{u}$.

At the end of each timestep, w is solved for using

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} = -\frac{\partial \omega_y}{\partial x} + \frac{\partial^2 v}{\partial y \partial z} \qquad (3)$$

then u is recovered directly from the continuity equation,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}. \qquad (4)$$

The remaining components of vorticity, ω_x and ω_z, are recovered from their definitions.

NUMERICAL METHOD

Spatial Discretisation

The numerical method is hybrid, in that the solution domain in the three directions is spatially discretised in three different ways. The solution domain is shown in Figure 1. It has finite length in the streamwise and spanwise directions, and is semi-infinite in the wall normal direction. In the streamwise direction, high-order compact difference schemes are used
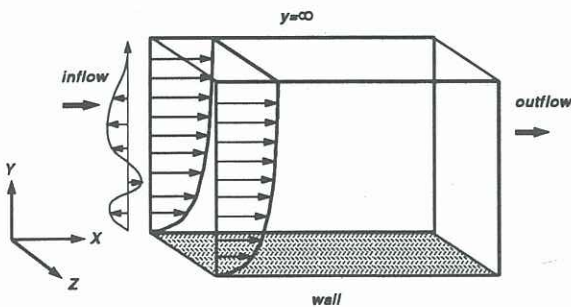

Figure 1: Diagram of the solution domain

for evaluating streamwise derivatives. The compact schemes (Lele (1992)) are sixth-order accurate with a five-point stencil, except at the boundary and near-to-boundary nodes:

The first derivative of a function $f(x)$ defined on a grid $x_i = i\Delta x$, $i = 0 \ldots N$ and with $f_i = f(x_i)$ is given by

$$\alpha f'_{i+1} + f'_i + \alpha f'_{i-1} =$$
$$\frac{1}{\Delta x}\left(\frac{a}{2}(f_{i+1} - f_{i-1}) + \frac{b}{4}(f_{i+2} - f_{i-2})\right) \quad (5)$$

with $\alpha = 1/3$, $a = 14/9$ and $b = 1/9$.

For the second derivative, the scheme is

$$\alpha f''_{i+1} + f''_i + \alpha f''_{i-1} =$$
$$\frac{1}{\Delta x^2}(a(f_{i+1} - 2f_i + f_{i-1}) +$$
$$\frac{b}{4}(f_{i+2} - 2f_i + f_{i-2})) \qquad (6)$$

with $\alpha = 2/11$, $a = 12/11$ and $b = 3/11$.

Derivatives at all grid points are calculated by applying these equations at each grid point, then solving the resultant tridiagonal system of equations. The fourth-derivative in the $\nabla^4 v$ term in (1) is evaluated by two applications of (6).

In the wall-normal direction, the spectral method of Spalart et al. (1991) based on the $(0, 1)$ Jacobi polynomials, $P_n^{(0,1)}(\xi)$, is used. The Jacobi polynomials, defined on the domain $\xi \in [-1, 1]$ are mapped to the domain $y \in [0, \infty)$ using the exponential mapping

$$\begin{aligned} \xi &= 2\eta - 1, \\ \eta &= e^{-\frac{y}{y_o}} \end{aligned} \qquad (7)$$

where y_o is a mapping parameter.

The expansion functions based on the Jacobi polynomials are chosen to automatically satisfy the boundary conditions. Three sets of polynomials are required to expand the components of velocity and vorticity in the wall-normal direction; those that have zero, one and two zeroes at the wall. We call these the f_n, g_n and h_n polynomials respectively, and their definitions are

$$\begin{aligned} f_n(y) &= \eta P_n^{(0,1)}(2\eta - 1), \\ g_n(y) &= (1 - \eta)\eta P_n^{(0,1)}(2\eta - 1), \\ h_n(y) &= (1 - \eta)^2 \eta P_n^{(0,1)}(2\eta - 1) \end{aligned} \qquad (8)$$

u, w, and ω_y are expanded in terms of the g_n polynomials. v, by continuity, has two zeroes at the wall, and therefore uses the h_n polynomials. ω_x and ω_z use the f_n polynomials. In all three cases, the function and all of its derivatives go to zero as $y \to \infty$.

In the spanwise direction, periodicity is assumed, and so a Fourier spectral method is used.

$$\phi(z) = \sum_{k=-\frac{nz}{2}}^{\frac{nz}{2}-1} \tilde{\phi}_k e^{i\frac{2\pi k}{L_z}z} \qquad (9)$$

In summary, a component of velocity and vorticity ϕ can be expanded in the form:

$$\phi(x, y, z, t) =$$
$$\sum_{k=-\frac{nz}{2}}^{\frac{nz}{2}-1} \sum_{j=1}^{ny} \tilde{\phi}_{jk}(x_i, t)\psi_j(y)e^{il_k z} \quad (10)$$

where $x_i = x_o + i\Delta x$, $i = 0 \ldots nx$ and $\psi_j(y)$ is either $f_j(y)$, $g_j(y)$ or $h_j(y)$.

The unknowns to be solved for at each timestep are the $nx \times ny \times nz$ expansion coefficients, $\tilde{\phi}_{jk}(x_i)$. A Galerkin statement is applied to each of the governing equations to generate, in terms of the expansion coefficients, ordinary differential equations for $\nabla^2 v$ and ω_y and algebraic equations for the other quantities.

Temporal Discretisation

Time-stepping of the O.D.Es is fully explicit, and a second-order Adams Bashforth method is used for temporal discretisation.

Extra Functions

In the wall-normal direction, extra functions are added to the expansions of the $u-$ and $v-$velocity components to ensure they both asymptote to their respective freestream values at the edge of the boundary layer. The modified expansions of u and v are

$$u = U_\infty(1-\eta) + \sum_{j=1}^{ny} \hat{u}_j g_j(y)$$
$$v = \hat{v}_x(1-\eta)^2 + \sum_{j=1}^{ny-1} \hat{v}_j h_j(y) \quad (11)$$

U_∞ can be a function of x, and is specified. On the other hand, outflow at the edge of the boundary layer, v_x, is not known a priori, and must be calculated as part of the solution.

Outflow/Inflow Boundary Conditions

To avoid spurious reflections of outgoing waves back into the domain at the outflow boundary, a non-reflective boundary condition needs to be applied. In this case, all variables are forced to satisfy the convective outflow condition

$$\frac{\partial \phi}{\partial t} = -c\frac{\partial \phi}{\partial x} \quad (12)$$

where c is the speed of the outgoing wave.

The direct simulation of a turbulent flow begins by specifying as an inflow boundary condition an unstable laminar flow. This unstable flow, which is then allowed to evolve in time into a fully turbulent flow, is obtained from Linear Stability Theory. In the case of incompressible boundary layer flow, linear stability is governed by the Orr-Sommerfeld equation:

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi =$$
$$\frac{1}{\alpha R}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi) \quad (13)$$

The disturbance streamfunction is given by $\psi(x, y, t) = \phi(y)e^{i(\alpha x - \beta t)} = \phi(y)e^{i\alpha(x-ct)}$ where α is the wavenumber, β is the frequency and $c = \beta/\alpha$ is the phase velocity of the disturbance. $U = U(y)$ is the mean velocity profile, R is the Reynolds number based on boundary layer displacement thickness and differentiation, denoted by $'$, is with respect to y, the direction normal to the surface. This equation is a dispersion relation between ϕ, α and c

To satisfy boundary conditions, ϕ is expanded in terms of the $h_n(y)$ polynomials. The Galerkin spectral method described above is used to discretise this equation, and an eigenvalue/eigenfunction problem results which is solved numerically using a QR algorithm. Given a mean flow profile (a Blasius profile is used), a Reynolds number and a complex streamwise wavenumber (α_r, α_i), the solution is a set of complex phase velocities (c_r, c_i) and complex disturbance eigenfunctions, from which profiles of the disturbance velocities can be generated.

For a spatially-evolving boundary layer the disturbance frequency β will be real and the corresponding complex wavenumber α needs to be calculated. This is accomplished using a Newton-Rhapson method and the Orr-Sommerfeld solver to find the complex α which corresponds to the given real β.
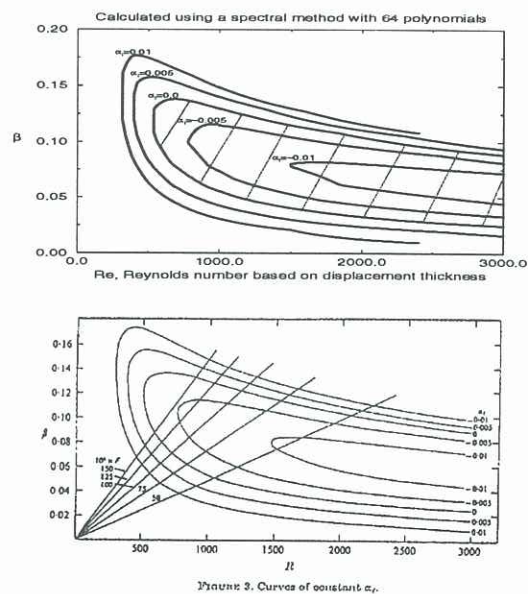

Figure 2: Comparison of computed lines of constant α_i (above) and Schlichting's (1960) results (below)

RESULTS

Orr-Sommerfeld Solver

The Orr-Sommerfeld solver was used to produce Figure 2. In this figure, disturbances with β, R parameters which lie in the shaded region (i.e. $\alpha_i < 0$)

are unstable. For a given simulation Reynolds number, the range of β corresponding to unstable disturbances can be determined. Good agreement with Schlichting's (1960) published results are evident.

Figure 3. shows the good agreement between the Orr-Sommerfeld eigenfunctions computed using the present method and Jordinson's (1970) published values. There is only a small variation in the computed eigenfunction as the mapping parameter, y_o, changes from 4.0 to 8.0.
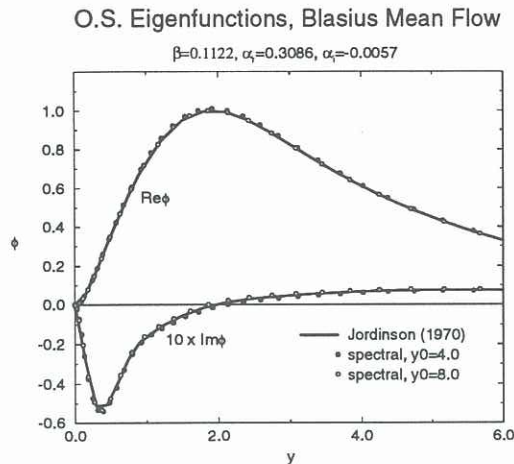
O.S. Eigenfunctions, Blasius Mean Flow

Figure 3: Orr-Sommerfeld eigenfunctions

Steady Navier-Stokes Solver

In Figure 4, results are presented from a computation in which the inflow boundary condition was a steady Blasius profile, and the simulation was continued until a steady-state was reached. In this way, the spatial development of the boundary layer profiles was captured. Comparison is made between u-profiles at several different x-locations and the Blasius profile. The y co-ordinate is normalised with respect to $(R_{\delta_i^*}/x)^{\frac{1}{2}}$ where $R_{\delta_i^*}$ is the Reynolds number based on displacement thickness at the inflow boundary. Agreement is excellent as the Blasius equation is a reasonable approximation to the steady Navier-Stokes equations at the Reynolds numbers considered.

CONCLUSION

A mixed spectral/compact difference numerical method has been developed to simulate the spatial evolution of a boundary layer. By casting the governing equations in velocity-vorticity form, complications and splitting errors associated with the calculation of pressure are avoided. To prevent the reflection of outgoing waves back into the solution domain, convective boundary conditions are imposed at the outflow boundary. The numerical method has been used to successfully compute solutions to the Orr-Sommerfeld
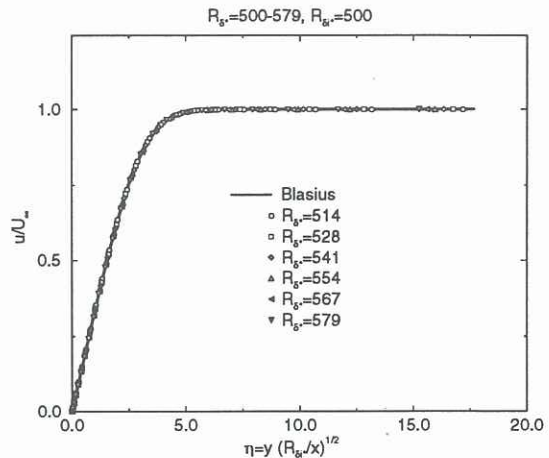

Figure 4: Steady-flow solutions to the Navier-Stokes equations

equation and a steady solution to the Navier-Stokes equation. The latter shows good agreement with Blasius' theory.

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