Twelfth Australasian Fluid Mechanics Conference
The University of Sydney, Australia 1995

UNSTEADY HIGH REYNOLDS NUMBER FLOW

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ABSTRACT

This paper examines early time solutions of the flow of a homogeneous viscous fluid around an impulsively started cylinder, within the context of laminar boundary layer theory. In particular, the finite time breakdown of boundary-layer theory for separating flows is examined. It is demonstrated that inclusion of higher-orders terms into the boundary-layer formulation can overcome this difficulty, as well as providing insight into reasons for such a breakdown.

INTRODUCTION

It has long been known that the boundary-layer equations provide a good approximation to laminar flow when flow separation is not present. In the limit of infinite Reynolds number, incompressible fluid flow can often be broken down into a large region of in-viscid irrotational flow, accompanied by a 'thin' boundary-layer containing all of the vorticity. The aim is then to match the flow in the boundary-layer with the irrotational flow outside.

There are some situations, however, where this procedure fails, generally due to a local violation of the assumptions of the boundary-layer equations. The unsteady flow around an impulsively started cylinder is one such example, since the boundary-layer equations are known to produce solutions that "breakdown" at a finite (non-dimensional) time of around $t = 3.0$ (Van Dommelen & Shen, 1980). This breakdown is indicated by infinite values of velocity and displacement thickness within the boundary-layer.

As the boundary-layer equations can be viewed as a reduction of the Navier-Stokes equation in the limit of infinite Reynolds number, previous techniques for dealing with this singularity involve the ad-hoc inclusion of finite Reynolds number terms into the boundary-layer equations. One strategy that has been implemented (Henkes & Veldman, 1987; Riley & Vasantha, 1989) is the incorporation of an interaction law into the boundary conditions, in order to account for the effect of a thickening boundary-layer on the external (irrotational) solution. While this technique appeared to suppress the singularity, theoretical work of Smith (1988), and Tutty & Cowley (1986) has shown that this type of interaction law will break down at a finite time for any Reynolds number.

The alternate strategy taken in this work is to look at the inclusion of other higher-order terms into the boundary-layer equations themselves. The term chosen permits pressure to vary across the boundary-layer, something that is not true of the traditional boundary-layer approximation.

GOVERNING EQUATIONS

The Helmholtz vorticity equation for non-dimensionalised two dimensional flow, relative to the cylinder, in cylindrical co-ordinates, can be written as

$$\frac{\partial \bar{\zeta}}{\partial \bar{t}} + u_r \frac{\partial \bar{\zeta}}{\partial r} + \frac{u_\theta}{r}\frac{\partial \bar{\zeta}}{\partial \theta} = \frac{1}{Re}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{\zeta}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \bar{\zeta}}{\partial \theta^2}\right) \quad (1)$$

where

$$Re = \frac{Ud}{\nu} \quad (2)$$

for the polar velocity components (u_r, u_θ), vorticity $\bar{\zeta}$, cylinder velocity U, cylinder diameter d, and kinematic viscosity ν.

It is convenient to consider the flow based on the reference frame of the cylinder, which leads to boundary conditions, $\mathbf{u} \rightarrow (U\cos\theta, U\sin\theta)$ as $r \rightarrow \infty$, as well as no slip conditions on the boundary.

In order to satisfy the continuity equation for a homogeneous flow, a stream function $\bar{\psi}$ can be defined such that

$$u_r = -\frac{1}{r}\frac{\partial\bar{\psi}}{\partial\theta}, \quad u_\theta = \frac{\partial\bar{\psi}}{\partial r} \tag{3}$$

$$\bar{\zeta} = \nabla^2\bar{\psi} \tag{4}$$

Introducing the classical boundary-layer scalings for a length scale $\epsilon = 1/\sqrt{Re}$, we can define

$$r = 1+\epsilon y, \ \theta = \pi - x, \ \bar{t} = 2t, \ u_r = 2\epsilon v, \ u_\theta = -2u \tag{5}$$

so that rescaled stream function ψ and vorticity ζ are given as

$$\zeta = \epsilon\bar{\zeta}, \quad \psi = \bar{\psi}/\epsilon \tag{6}$$

With these scalings, the governing equations (1) and (4) become

$$\begin{aligned}\frac{\partial\zeta}{\partial t} \quad + \quad & \frac{u}{1+\epsilon y}\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} = \\ & \frac{1}{1+\epsilon y}\frac{\partial}{\partial y}\left((1+\epsilon y)\frac{\partial\zeta}{\partial y}\right) \\ & + \frac{\epsilon^2}{(1+\epsilon y)^2}\frac{\partial^2\zeta}{\partial x^2}\end{aligned} \tag{7}$$

with

$$u = \frac{\partial\psi}{\partial y}, \quad v = \frac{1}{1+\epsilon y}\frac{\partial\psi}{\partial x} \tag{8}$$

and

$$\zeta = \frac{\partial^2\psi}{\partial y^2} + \frac{\epsilon}{1+\epsilon y}\frac{\partial\psi}{\partial y} + \frac{\epsilon^2}{(1+\epsilon y)^2}\frac{\partial^2\psi}{\partial x^2} \tag{9}$$

From the assumption of potential flow outside the boundary-layer gives the boundary condition $\partial\psi/\partial y \rightarrow \sin x$ as $y \rightarrow \infty$. We are free to choose $\psi(x,0,t) = 0$, and by assuming symmetry about $y = 0$, $\psi(0,y,t) = \psi(\pi,y,t) = 0$. The no-slip condition on the surface requires that $\partial\psi/\partial y$ on $y = 0$

By considering the limit as $\epsilon \rightarrow 0$, equations (7), (8), and (9) yield the classical boundary-layer equations, in vorticity form

$$\frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} = \frac{\partial^2\zeta}{\partial y^2} \tag{10}$$

where

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x} \tag{11}$$

and

$$\zeta = \frac{\partial^2\psi}{\partial y^2} \tag{12}$$

Given the limitation of the classical boundary-layer equations for separating flows, we next consider ways of overcoming these shortcomings. Cowley, Van Dommelen & Lam (1990) have argued that since the boundary-layer formulation does not allow pressure to vary across the layer, and that viscosity is also neglected in this direction, there is no force to oppose the ejection of fluid from the boundary-layer. With this in mind, we consider the addition of a higher-order term $(\epsilon^2(\partial^2\psi)/(\partial x^2))$ in (8) into the stream function-vorticity equation, allowing the pressure to vary across the boundary-layer, so that the stream function is related to vorticity as

$$\zeta = \frac{\partial^2\psi}{\partial y^2} + \epsilon^2\frac{\partial^2\psi}{\partial x^2} \tag{13}$$

NUMERICAL TECHNIQUE

Initially, when u_r and u_θ are instantaneously set to irrotational flow everywhere, the vorticity equation (10) simplifies to the heat equation. The analytical solution of this predicts that initially vorticity will be infinite on the boundary, in the limit as $t \rightarrow \infty$, but this singularity can be resolved by the introduction of suitable variables.

$$Z = \zeta\sqrt{t}, \quad Y = \frac{y}{\sqrt{t}}, \quad \Psi = \frac{\psi}{\sqrt{t}} \tag{14}$$

This solution also suggests that grid stretching should be performed in the radial direction, and an exponential stretching is used to achieve the required resolution near the inner boundary. The outer boundary is truncated at a finite distance, typically $Y = 30$, although results are seen to be relatively insensitive to this parameter, provided it is large.

Since we expect singular behaviour within the boundary-layer, the grid stretching of Riley & Vasantha (1989) is introduced, which allows concentration of the grid in the x direction towards regions of sharp vorticity change.

The numerical solution of equation (10) for $t > 0$ proceeds with use of centered differences on all terms. Implicit time stepping is used, with successive approximations being used on non-linear terms at each time step. For the classical boundary-layer equations, the implicit terms were solved using tridiagonal elimination, while the extended equations required the use of a block tridiagonal scheme to solve for the stream function.

RESULTS

In figure (1) the boundary-layer displacement thickness, $\delta(x) = \int_0^\infty (1 - u(x,y)/\sin x)\, dy$, is presented. This quantity is useful, as it provides comparison with previously mentioned work on boundary-layer break down. From these results, it is clear that the boundary-layer is quickly growing towards a very large displacement thickness, at around the time and position predicted by VanDommelen & Shen, of $x_s = 1.937$ and $t_s = 3.00$.

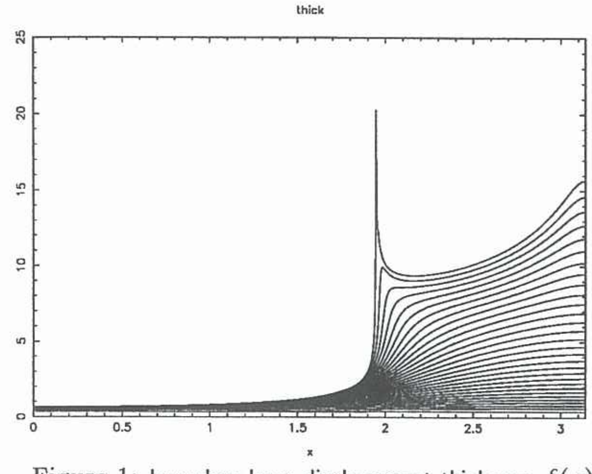

Figure 1: boundary-layer displacement thickness, $\delta(x)$, for $Re \rightarrow \infty$, $0 \leq t \leq 3.0$, $\delta t = 0.1$

The next case considered is that were the pressure is premitted to vary across the boundary-layer, by solving equation (13) for stream function. In figure (2) demonstrates how the singularity is diminished with decreasing Reynolds number. It is evident that the singularity encountered in the infinite Reynolds number case is a result of the large ejection of vorticity from withing the boundary layer, occuring at finite Reynolds number. As this parameter is increased, the erruption becomes more sudden, an thus harder to track numerically.

In the stream function, this process is evident function as the splitting of the large recirculation region, into smaller circulation regions, a phenomenon predicted by the numerical and experimental work of Ta Phuoc Loc & Bouard (1985), as well as others.

Simulations were also performed on the full boundary-layer equation, given by (7), (8) and (9). While there are some changes in the detail of the flow, the general sequence of events is maintained.

CONCLUSION

In paper demonstrates that by modifying the classical boundary-layer equation, to take into account the effect of varying pressure across the layer, it is possible to evolve an impulsively-started boundary-layer flow well past times possible in classical boundary-layer theory. Insight is also provided into how boundary-layer theory can break down, and how this is related to the evolution of boundary layers.

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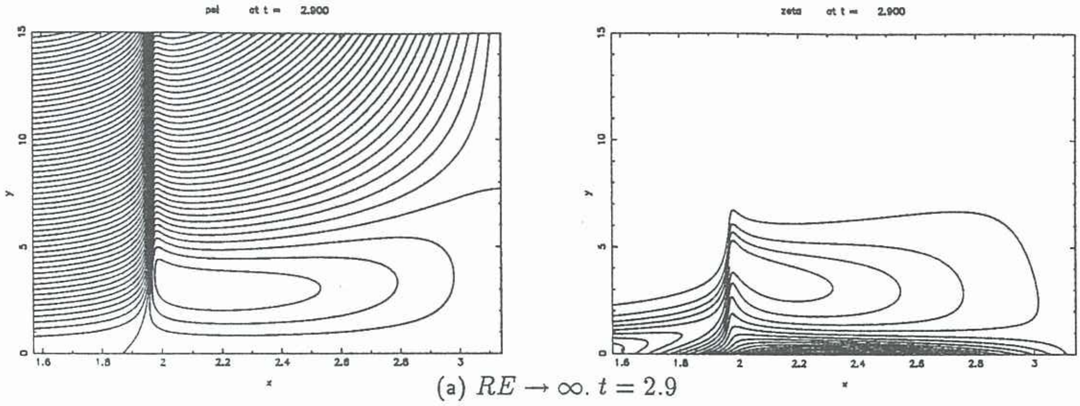
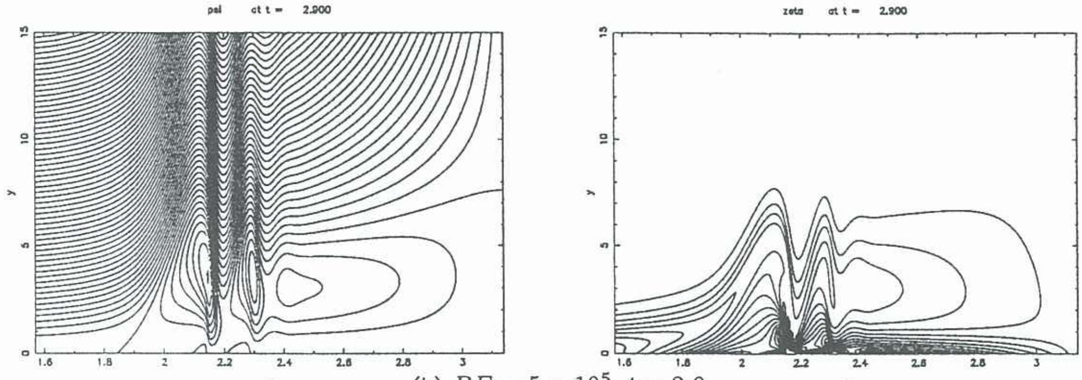
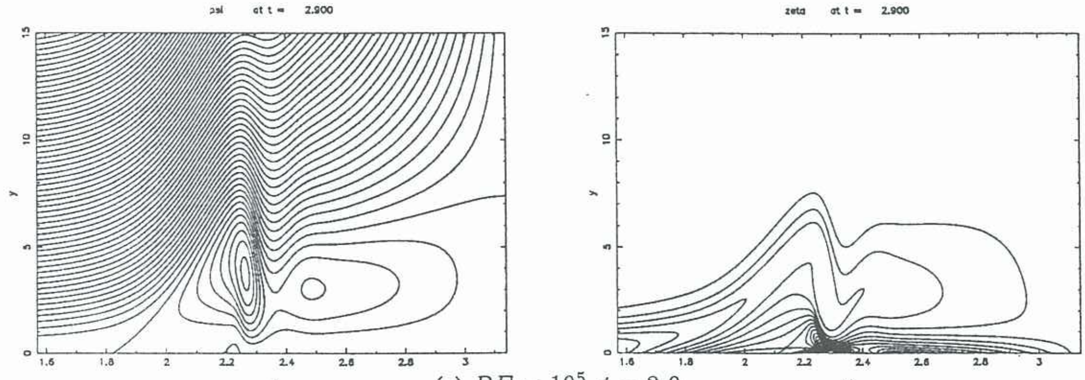


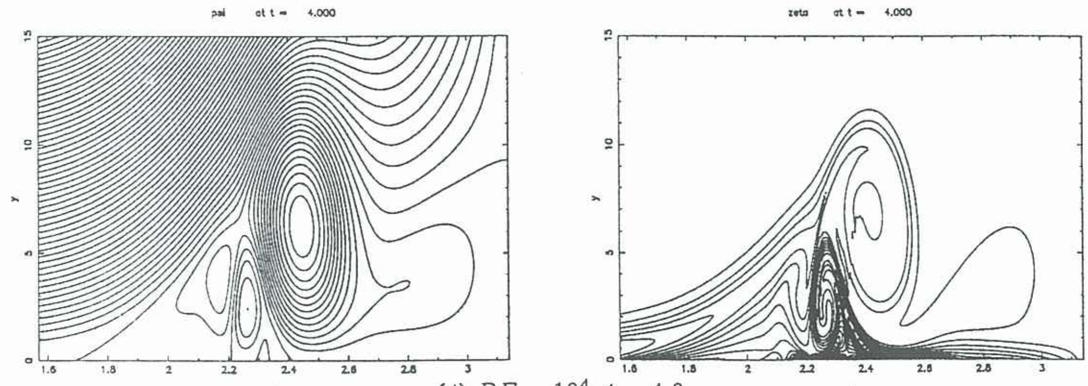
(a) $RE \to \infty$. $t = 2.9$



(b) $RE = 5 \times 10^5$. $t = 2.9$



(c) $RE = 10^5$. $t = 2.9$



(d) $RE = 10^4$. $t = 4.0$

Figure 2: Stream Function and Vorticity. $-20 \leq \psi \leq 5$. $\Delta\psi = 0.25$. $-3 \leq \zeta \leq 2$. $\Delta\zeta = 0.1$