

## TWO-DIMENSIONAL LINEAR STABILITY ANALYSIS OF THE BOUNDARY LAYER IN A DIFFERENTIALLY HEATED CAVITY

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### ABSTRACT

The non-parallel effects on the stability of the vertical boundary layer in a differentially heated cavity are investigated by incorporating streamwise variation into the linear stability equations. Such an analysis gives rise to growth rates and wavenumbers that are highly dependent on the transverse location and the disturbance variable under consideration. This is in contrast with the parallel stability analysis generally applied to such flows in which the growth rates and wavenumbers are independent of transverse location and disturbance variable. A direct stability analysis is also performed by integration of the complete Navier-Stokes equations. The stability properties are obtained by introducing an oscillatory heat input at the upstream end of the boundary layer. The non-parallel linear stability analysis is shown to be in good agreement with the direct stability analysis.

### INTRODUCTION

The flow configuration is a square cavity of width,  $L$ , with isothermal vertical side walls at temperatures,  $T_h$  and  $T_c$  and adiabatic top and bottom walls. The flow is governed by the two-dimensional Navier-Stokes equations under the Boussinesq assumption and by choosing length, velocity and temperature scales,  $L$ ,  $\nu/L$  and  $\Delta T = (T_h - T_c)/2$  these form the set of non-dimensional equations,

$$U_x + V_y = 0, \quad (1)$$

$$U_t + UU_x + VU_y = -P_x + U_{xx} + U_{yy} + \frac{Ra}{Pr}T, \quad (2)$$

$$V_t + UV_x + VV_y = -P_y + V_{xx} + V_{yy}, \quad (3)$$

$$T_t + UT_x + VT_y = \frac{1}{Pr}(T_{xx} + T_{yy}) + S. \quad (4)$$

Previous investigators have studied the large time behaviour for his system over a range of  $Ra \equiv g\beta\Delta TL^3/\nu\kappa$  and  $Pr \equiv \nu/\kappa$  and it is known to undergo a transition from steady laminar flow to unsteady turbulent flow (Paolucci and Chenoweth, 1989, Janssen and Henkes, 1995). However, for  $Ra = 6 \times 10^8$  and  $Pr = 7.5$ , used throughout this study, the flow is steady. Perturbations are introduced into the steady state system for the direct stability analysis. The numerical solution to the steady state system is used as the baseflow for the non-parallel linear stability analysis.

### DIRECT STABILITY ANALYSIS

The non-dimensionalised Navier-Stokes equations are solved numerically using an implicit second order time integration and a finite volume spatial discretization on a non-staggered mesh. The formulation is the same as that used in Patterson and Armfield (1990) and further details are provided in Armfield (1991). A time step of  $\Delta t = 5 \times 10^{-7}$  and a  $120 \times 120$  grid is used in which the grid spacing increases with distance away from the boundaries.

Figure 1 shows the resulting steady state temperature and streamfunction fields,  $(\bar{T}, \bar{\psi})$ , with the hot wall on the left side of the cavity. The flow exhibits thin vertical boundary layers on the hot and cold walls and the interior is vertically stratified. The perturbation is then introduced as a sinusoidal heat input at the base of the hot boundary layer. Defining the origin to be at the base of the hot wall, the source term,  $S$ , is non-zero only in the region,  $0 < x < 0.01$ ,  $0 < y < 0.01$  where,

$$S = A \sin(2\pi f_i t). \quad (5)$$

After an initial transient phase, where the distur-

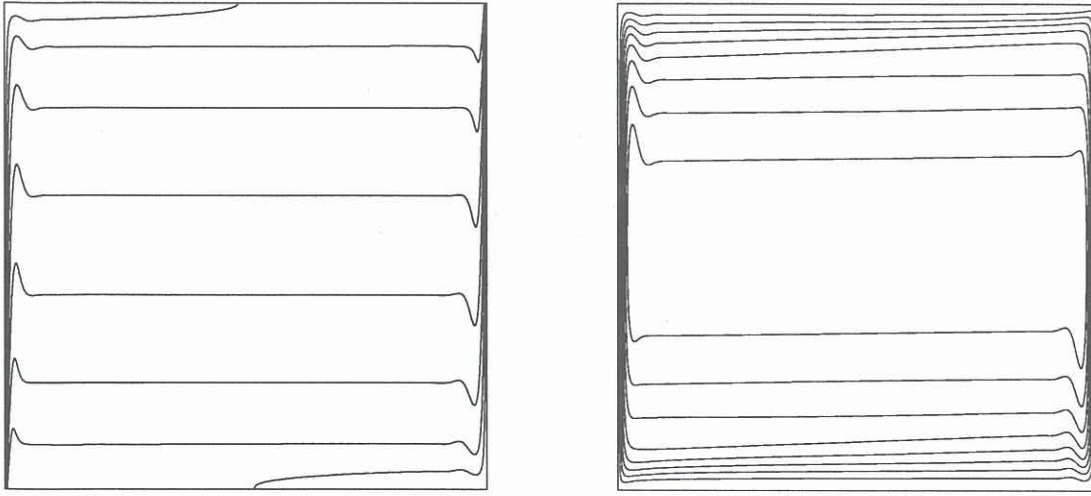


Figure 1: (a) The temperature,  $\bar{T}$ , contours at steady state. (b) The streamfunction,  $\bar{\psi}$  contours at steady state.

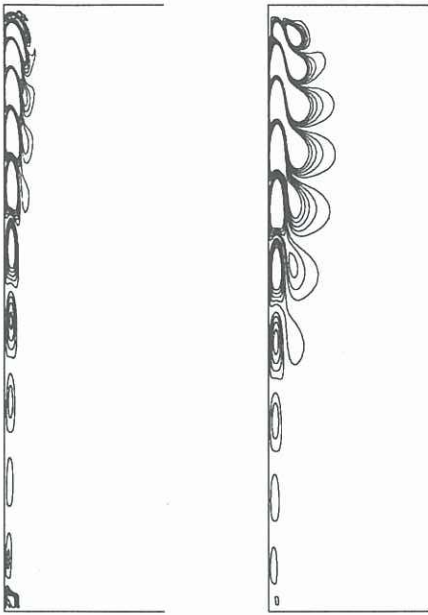


Figure 2: (a) The temperature perturbation,  $T - \bar{T}$  and (b) the vertical velocity perturbation,  $U - \bar{U}$ .

balance spreads up the boundary layer, a steady pattern of oscillations is established. Shown in figure 2 are the instantaneous temperature and vertical velocity perturbation fields resulting from a perturbation of frequency  $f_i = 4000$ . The perturbation is rapidly damped once it reaches the top boundary and is not carried across to the opposite cold wall. This was the case for all frequencies considered in this study.

Direct stability analysis using the complete Navier-Stokes equations allows both nonparallel and nonlinear effects to be investigated. Here, the disturbance amplitude  $A$  was chosen such that the resulting perturbations remained linear. Linearity was tested by ensuring that the wave growth rate was insensitive to  $A$  and a value of  $A = 0.1$  was used throughout this study.

## TWO-DIMENSIONAL STABILITY ANALYSIS

Numerous linear stability studies of vertical natural convection boundary layers have been performed following the first formulation of the problem by Plapp (1957). The linear stability equations for the perturbation streamfunction and temperature,  $(\psi', T')$  given a baseflow,  $(\bar{\psi}, \bar{T})$  are formed by substituting  $(\bar{\psi} + \psi', \bar{T} + T')$  into the Navier-Stokes equations and then eliminating the non-linear terms. Typically these analyses have used baseflows attained from similarity solutions using the boundary layer assumptions. A parallel flow approximation is then made and the non-parallel baseflow terms are neglected in the stability equations. The perturbations are then assumed to take the form,

$$(\psi', T') = (\psi(y), T(y)) \exp(kx - i\omega t). \quad (6)$$

Here, the non-parallel stability equations are formulated by including a first order correction to the parallel flow assumption whereby the first derivative in the streamwise direction of the baseflow and the perturbation eigenfunction are not neglected. Furthermore, the baseflow used is the numerical solution to the full Navier-Stokes equations and thus the stability results are directly comparable to the direct stability analysis. The baseflow streamfunction and temperature fields, at a given height  $x_0$ , are assumed to have the form,

$$\bar{\psi}(x, y) = \bar{\psi}(x_0, y) + \xi \bar{\psi}_x(x_0, y), \quad (7)$$

$$\bar{T}(x, y) = \bar{T}(x_0, y) + \xi \bar{T}_x(x_0, y), \quad (8)$$

where  $\xi = x - x_0$  and the subscripts refer to differentiation. The perturbation is represented by,

$$\psi'(x, y, t) = (\phi_0(y) + \xi \phi_1(y)) \exp(kx - i\omega t), \quad (9)$$

$$T'(x, y, t) = (T_0(y) + \xi T_1(y)) \exp(kx - i\omega t). \quad (10)$$

Hence, at any given location  $x = x_0$  we seek constant frequency solutions where  $\omega$  is the angular frequency,  $\Re(k)$  is the spatial amplification and  $\Im(k)$  is



the wavenumber. Substituting equations (7-10) into the Navier-Stokes equations gives the coupled set of eigenvalue equations,

$$L_{0,1}\psi_0 + L_{0,2}T_0 + L_1\psi_0 + L_2\psi_1 = 0, \quad (11)$$

$$\begin{aligned} M_{0,1}\psi_0 + M_{0,2}T_0 + M_{1,1}\psi_0 + M_{1,2}T_0 + \\ M_{2,1}\psi_1 + M_{2,2}T_1 = 0, \end{aligned} \quad (12)$$

$$L_{0,1}\psi_1 + L_{0,2}T_1 + L_1\psi_1 + L_3\psi_0 + L_4\psi_1 = 0, \quad (13)$$

$$\begin{aligned} M_{0,1}\psi_1 + M_{0,2}T_1 + M_{1,1}\psi_1 + M_{1,2}T_1 + \\ M_{3,1}\psi_0 + M_{3,2}T_0 + M_{4,1}\psi_1 + M_{4,2}T_1 = 0. \end{aligned} \quad (14)$$

The differential operators are given below, in which  $D$  refers to differentiation with respect to  $y$ .

$$\begin{aligned} L_{0,1} &= -\frac{1}{Gr}((D^2 + k^2)^2) + \\ &(\bar{\psi}_y k - i\omega)(D^2 + k^2) - \bar{\psi}_{yyy}k, \end{aligned}$$

$$L_{0,2} = -\frac{1}{Gr}D,$$

$$L_1 = \bar{\psi}_{xyy}D - \bar{\psi}_x(D^3 + k^2D),$$

$$\begin{aligned} L_2 &= -\frac{1}{Gr}(4kD^2 + 4k^3) + \bar{\psi}_y(D^2 + 3k^2) \\ &- 2ik\omega - \bar{\psi}_{yyy} - \bar{\psi}_x 2kD, \end{aligned}$$

$$L_3 = \bar{\psi}_{xy}(kD^2 + k^3) - \bar{\psi}_{yyyx}k,$$

$$L_4 = \bar{\psi}_{xy}(D^2 + 3k^2) - \bar{\psi}_{yyyx},$$

$$M_{0,1} = -\bar{T}_y k,$$

$$M_{0,2} = -\frac{1}{GrPr}(k^2 + D^2) - i\omega + k\bar{\psi}_y,$$

$$M_{1,1} = \bar{T}_x D, \quad M_{1,2} = -\bar{\psi}_x D,$$

$$M_{2,1} = -\bar{T}_y, \quad M_{2,2} = -\frac{1}{GrPr}2k + \bar{\psi}_y,$$

$$M_{3,1} = -\bar{T}_{xy}k, \quad M_{3,2} = \bar{\psi}_{xy}k,$$

$$M_{4,1} = -\bar{T}_{xy}, \quad M_{4,2} = \bar{\psi}_{xy}.$$

The linear stability equations have been non-dimensionalised using a more general scaling than is used for the cavity. An arbitrary length scale,  $\delta$  and a velocity scale,  $g\beta\Delta T\delta^2/\nu$  are used and  $Gr = g\beta\Delta T\delta^3/\nu^2$ . The equations (11-14) are solved with homogeneous boundary conditions for  $\phi_0$ ,  $\phi_{0y}$ ,  $\phi_1$ ,  $\phi_{1y}$ ,  $T_0$  and  $T_1$  at  $x = 0$  and  $x = x_{max}$ . The outer boundary condition is chosen to reduce computational time while maintaining accuracy and  $x_{max}$  of between 0.1 and 0.2 was generally used. The numerical solution procedure is a straightforward shooting method with orthonormalisation (Davey, 1973).

## RESULTS

The results at a height of  $x = 0.2$  are first examined. Figure 3(a-b) shows the baseflow temperature

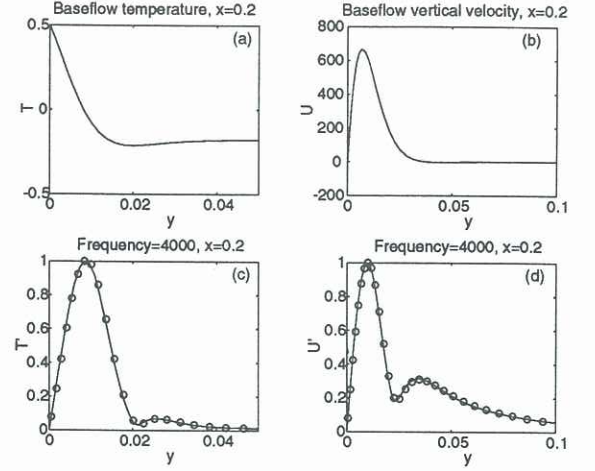


Figure 3: The steady state baseflows for the temperature (a) and vertical velocity (b). The temperature eigenfunction,  $T_0$  (solid line) and the numerical eigenfunction  $A_T$  (circles) (c) and the vertical velocity eigenfunction,  $\phi_{0y}$  (solid line) and the numerical eigenfunction  $A_U$  (circles) (d). All quantities are for  $x=0.2$ .

and vertical velocity at  $x = 0.2$  which results from the steady state numerical simulation. A perturbation was imposed at the base of the hot boundary layer with a frequency,  $f_i = 4000$  and the amplitude,  $A_p(x, y)$ , of the perturbation signal observed in a disturbance variable,  $p$ , was calculated. For example,  $A_T(0.2, y)$  is the amplitude of the temperature disturbance at a height  $x = 0.2$ . This quantity can be compared to the absolute value of the eigenfunction from the two-dimensional stability analysis. In figure 3(c) the eigenfunction,  $T_0(0.2, y)$ , and the amplitude,  $A_T(0.2, y)$ , each normalised by their maxima, are shown. The agreement between the non-parallel analysis and the direct stability analysis is very good. Similar agreement is also seen between the eigenfunction for the vertical velocity,  $\phi_{0y}$  and  $A_U(0.2, y)$  in figure 3(d).

The amplification seen in any flow variable  $p$ , defined as  $a = (\partial A_p(x, y)/\partial x)/A_p(x, y)$ , can be calculated using the direct stability analysis. Using the linear stability analysis the amplification is  $a = \Re(k) + \Re(p_1(y)/p_0(y))$ . Shown in figure 4 are the amplifications of the temperature signal for a range of frequencies and at heights,  $x = 0.2$  and  $x = 0.5$ . Also shown are the amplifications found using parallel assumptions in the linear stability analysis. In this case the stability equations reduce to,

$$L_{0,1}\psi_0 + L_{0,2}T_0 = 0, \quad (15)$$

$$M_{0,1}\psi_0 + M_{0,2}T_0 = 0. \quad (16)$$

Clearly, the solution using the parallel assumptions is unsatisfactory and even half way up the cavity, where the flow is almost parallel, there is large discrepancy between the parallel stability analysis and the direct

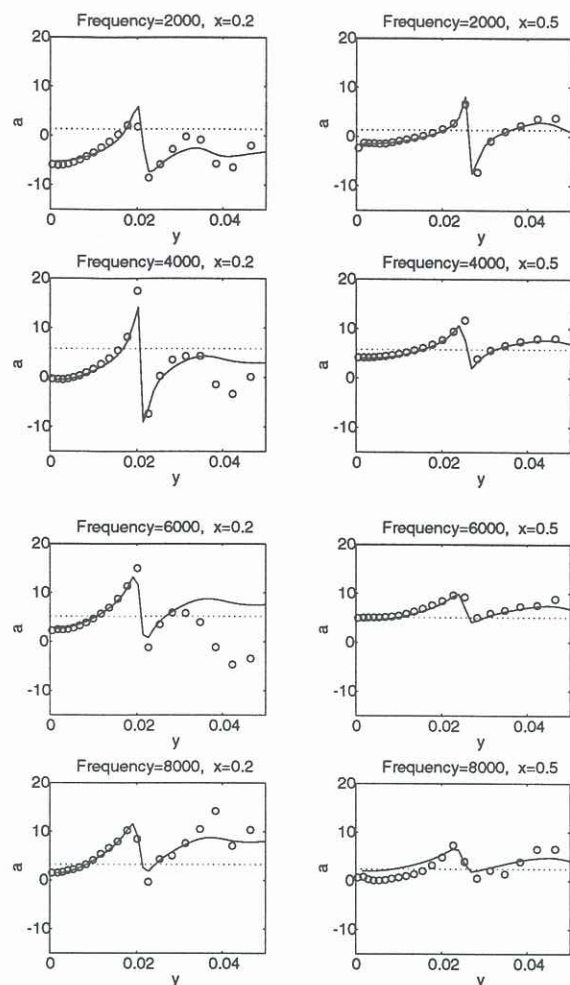


Figure 4: The amplification in the temperature signal using non-parallel assumptions,  $a = k_r + \Re(T_1/T_0)$  (solid lines), using parallel assumptions,  $a = k_r$  (dashed lines) and the numerical simulation,  $a = (\partial A_T / \partial x) / A_T$  (circles) at  $x=0.2$  and  $x=0.5$  for frequencies  $f_i = 2000, 4000, 6000, 8000$ .

stability analysis. The amplification is highly dependent on the transverse location and this is only modelled when the non-parallel effects are taken into account.

Estimates of the wavelength at certain distances along the wall can be made by measuring the peak-to-peak distances. In the two-dimensional linear analysis the wavenumber is also dependent on the transverse location and for the temperature signal the wavenumber is,  $K = \Im(k) + \Im(T_1/T_0)$ . Figure 5 shows the wavenumbers,  $K(y)$  at locations,  $x = 0.3$ ,  $x = 0.5$  and  $x = 0.7$  up the hot wall for  $f_i = 3000$ . Also shown are the estimates of the wavenumber determined by the peak-to-peak distance observed in the numerical simulation and the wavenumbers determined using parallel linear stability analysis. The non-parallel analysis predicts the wavenumber more accurately than the parallel analysis. The sharp dip

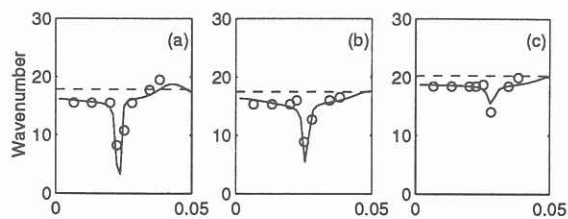


Figure 5: The temperature wavenumber,  $K = k_i + \Im(T_1/T_0)$  (solid lines), the temperature wavenumber using parallel assumptions (dashed lines) and the wavenumber determined from the direct analysis (circles) at locations (a)  $x=0.3$ , (b)  $x=0.5$  and (c)  $x=0.7$ .

in the wavenumber at around  $y = 0.025$  seen in the two-dimensional analysis is associated with a 180 degrees phase change in the temperature eigenfunction that occurs at this location and is a feature of the flow that the parallel analysis cannot predict.

## CONCLUSIONS

The non-parallel formulation for the linear stability equations has resulted in excellent agreement with the direct stability results. Although the parallel formulation predicted the amplification well at some cross-stream locations it performed poorly at others. Agreement between previous parallel analysis and experimental results may be reliant on the amplifications being measured at these points. The parallel analysis also overpredicted the wavenumber of the disturbance.

**Acknowledgements** Support for this work was provided through the provision of an Australian Postgraduate Research Award and the Samaha Research Scholarship.

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