

## TURBULENT DISPERSION WITH BROKEN REFLEXIONAL SYMMETRY

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### ABSTRACT

The dispersion of material from an instantaneous point release and a constant-gradient release in homogeneous turbulence is considered. Expanding on Taylor's (1921) approach, we allow for anisotropic flows with broken reflexion symmetry (i.e. lacking mirror symmetry; Batchelor, 1953). By considering Lagrangian trajectories in such flows and using appropriate high Reynolds-number models of them, significant qualitative and quantitative variations from Taylor's results are found. These include marked oscillations in the velocity autocorrelations, drastic reductions in turbulent dispersion, and spiralling of Lagrangian trajectories. As proposed by Moffatt (1983), the angular momentum of Lagrangian particles is a key parameter and 'skew-diffusion,' i.e. transport orthogonal to concentration gradients, is a prominent effect.

### INTRODUCTION

Taylor (1921) provides a simple and powerful exposition of transport in turbulent flows. He derived the dispersion (mean-square displacement) in terms of the velocity autocorrelation of Lagrangian trajectories. For simple isotropic, homogeneous and stationary flows, he further assumed a simple exponential function for the autocorrelation and obtained a comprehensive description of the dispersion. Since the transport of scalars is expected to be predominantly accomplished by the advective transport, Taylor's result gives a number of anticipated transport effects, particularly the eddy-diffusion character at large times.

More modern theories expand on Taylor's work by attempting to predict the velocity autocorrelation based on physical properties of turbulent flows, notably the small-scale acceleration field. Beginning with Obhukov (1959), through to Thomson (1987), Borgas & Sawford (1991, 1994) and Pope (1994), an approach has developed with stochastic models of the Lagrangian-velocity time series in the form of a Langevin equation, where it is understood that the time scales resolved are much greater than the Kolmogorov microscales where viscous effects are at work (Monin & Yaglom, 1975). The viscous effects manifest very large fluid-particle accelerations, correlated over very short times, so for sufficiently coarse times, the Lagrangian

properties are described as an approximate diffusion in velocity phase space. Simply with this knowledge, and the further specification of statistically homogeneous, isotropic and stationary flow fields, we obtain the *prediction* that the velocity autocorrelation is an exponentially decreasing function of time.

Despite the simplicity of this approach and the ease with which apparent generalisations to inhomogeneous turbulence may be made, the utility beyond the highly idealised example is unclear. This is mainly because no general unique diffusion process in velocity-position phase space can be written down with present knowledge. Thus to shed further light on the problems involved with such Lagrangian stochastic models, we examine the flow most minimally disturbed from the pure idealisation by retaining homogeneity but relaxing isotropy, but only to the extent that the flow lacks reflexion symmetry (Borgas *et al.*, 1995). This means that reflections of flow properties in planes normal to some direction  $\underline{\Omega}$  are statistically different. In the simplest terms, this means that on average flow trajectories spiral in either a left- or right-handed sense with respect to  $\underline{\Omega}$ . The turbulence field is actually axisymmetric with respect to rotations about  $\underline{\Omega}$  (Batchelor, 1953), but we add the extra restriction of 'equipartition' of the kinetic energy amongst the three orthogonal directions at any point, which is not strictly necessary for our purposes, but highlights the effects of the broken reflexion. Despite the 'equipartition', broken reflexion is a characteristic of the large energy-containing scales and, as smaller and smaller scales are examined, the effects of anisotropy are less and less important.

Recapping, we consider the dispersion due to a field of turbulent eddies of uniform average kinetic energy (per unit mass)  $\frac{3}{2}\sigma_u^2$  and for which the transfer of kinetic energy from some large-scale forcing mechanism to the viscous scales, where it is dissipated, is likewise uniform ( $=\bar{\epsilon}$  per unit mass). Two cases are contrasted: first, Taylor's case where the eddies have no preferred sense of rotation; second, when the eddies have some preferred sense of rotation with respect to the direction  $\underline{\Omega}$ . Examples of specific important predictions are: skew turbulent fluxes, where material is transported at right angles to the gradient of that material; spiralling fluid-particle trajectories; and drastic



reductions of transport orthogonal to the axis  $\underline{\Omega}$ , when the spiralling sense about this direction is strong. Central to these properties is the existence and size of the mean angular momentum of a fluid particle. This is an intrinsic Lagrangian property and may be non-zero even when the Eulerian flow statistics are independent of position, i.e. homogeneous. The Eulerian probability distribution for velocity at a point is a key property in the velocity phase-space diffusion process, but there is no information about angular momentum in it. Another characteristic of broken-reflexion is non-zero *helicity* (Moffatt & Tsinober, 1992), but here we make no use of that concept, which is essentially a two-point velocity statistic.

These results have practical importance in two senses. First, the idealisations reflect the simplest possible models of several real fluid flows: rotating turbulent fluid mass with axisymmetry about the rotation axis (Zeman, 1994); or with axisymmetry about the direction of an imposed magnetic field for the turbulent flow of electrically conducting fluid (Moffatt, 1983); or with the axisymmetry about the downstream direction in a wind tunnel with an array of right (or left) handed propellers acting as a grid (Kholmyansky *et al.*, 1991). Second, and more importantly, we arrive at a system where the diffusion process for velocity is not unique, but this non-uniqueness can be related to different physical situations (each characterised by a specific angular momentum) and not simply a mathematical artifact. Thus it is clear additional physical information is required in order to properly model the dispersion in complex flows.

### LAGRANGIAN STOCHASTIC MODELS

The velocity along a Lagrangian trajectory is modelled with the equation (Thomson, 1987)

$$du_i = a_i dt + \sqrt{C_0 \bar{\epsilon}} dW_i, \quad (1)$$

where the white noise  $dW$  reflects the diffusion-like character in velocity phase space and  $\underline{a}$  is a yet to be determined function of velocity. The coefficient  $C_0$  is a universal parameter which reflects the relative size of mean-square turbulent accelerations integrated over the typical duration of such impulses: a large number of such uncorrelated acceleration events leads to the diffusion-like nature. The value of  $C_0$  is around six or seven (Sawford, 1991).

The Lagrangian position of course follows from

$$dx_i = u_i dt, \quad (2)$$

and together with (1), the system is equivalent to the Fokker-Planck equation

$$\frac{\partial P}{\partial t} + u_i \frac{\partial P}{\partial x_i} = -\frac{\partial a_i P}{\partial u_i} + \frac{1}{2} C_0 \bar{\epsilon} \frac{\partial^2 P}{\partial u_i \partial u_i} \quad (3)$$

for the transition probability density,  $P$ , between initial and final states in velocity-position phase space. It is well known that the Eulerian probability density for velocity at a point  $\underline{x}$  also satisfies (3), which fixes some properties of  $\underline{a}$  (Thomson, 1987). For Gaussian one-point statistics, which are appropriate for the homogeneous case considered here, we have that

$$a_i = -\frac{1}{2} \frac{C_0 \bar{\epsilon}}{\sigma_u^2} u_i + \varepsilon_{ijk} \Omega_j u_k \quad (4)$$

where in general  $\underline{\Omega}$  is a function of the velocity magnitude (i.e. a non-linear equation (1)) but for our purposes we simply take  $\underline{\Omega}$  as a vector of unknown constant magnitude. It can be assumed that external symmetry clearly prescribes

the *direction* that  $\underline{\Omega}$  points. Finally, the alternating tensor  $\varepsilon_{ijk}$  is non-zero only when each of the integers  $i, j$  and  $k$  are distinct from one another, and has the value of plus or minus one for cyclic or acyclic permutations of  $\{1, 2, 3\}$ .

The physical processes embodied in (4) are that a Lagrangian particle changes its velocity over some short time predominantly due to random small-scale increments which are statistically isotropic (independent of direction) plus an isotropic drift opposed to the current velocity, which serves to relax the velocity back to zero and maintains constant kinetic energy, and lastly an anisotropic rotational change at right angles to the current velocity which does not change the velocity magnitude (i.e., the kinetic energy) but induces a spiral trajectory and angular momentum for the particle in the long run.

The stochastic modelling perspective is that (3) (with known Eulerian velocity distribution as input) gives a non-unique model, but which is hoped is constrained by some other statistical property. Our position is that model (4) represents different physical flows where  $\underline{\Omega}$  can be related to a physical property disconnected with the one-point Eulerian velocity distribution. There simply is not a unique model based solely on the Eulerian velocity statistics. This apparently subtle distinction is important because it is the first clear indication of the kind of extra information required. More general inhomogeneous anisotropic flows, say convective atmospheric boundary layers, are too complex to make much theoretical progress and generally more pragmatic solutions are attempted for such problems (Luhar & Sawford, 1995).

### SOLUTIONS

The linear equation is solved easily. Taylor's work indicates the importance of the velocity autocorrelation, but for our case we need a matrix of correlations,

$$R_{ij} = \langle u_i(t) u_j(0) \rangle,$$

therefore obtaining from (1)

$$\dot{R}_{ij} = A_{ik} R_{kj} \quad (5)$$

where  $A_{ij} = -\frac{1}{2} \frac{C_0 \bar{\epsilon}}{\sigma_u^2} \delta_{ij} + \varepsilon_{ikj} \Omega_k$ . The important things to note from this equation are that exponential autocorrelations clearly occur when  $\underline{\Omega}$  is omitted, but when it is not, there is coupling of both velocity components orthogonal to  $\underline{\Omega}$ . In fact, the velocity component parallel to  $\underline{\Omega}$  is not affected by the broken reflexion and the autocorrelation for this component is given by Taylor's exponential form.

We let  $t_L = \frac{2\sigma_u^2}{C_0 \bar{\epsilon}}$  be a Lagrangian time scale with which we measure time and use  $\sigma_u$  as a measure of velocity. Then for the rest of this paper we consider dimensionless variables relative to these scales and accordingly have a dimensionless measure of broken reflexion given by  $\omega = |\underline{\Omega}| t_L$ .

The fact that orthogonal components of velocity are coupled in the plane orthogonal to  $\underline{\Omega}$  inevitably means that (damped) oscillations occur in the autocorrelation. For the linear model we have

$$R_{\parallel} = e^{-t}, \quad (6)$$

for the autocorrelation parallel to  $\underline{\Omega}$ ,

$$R_{\perp} = e^{-t} \cos \omega t, \quad (7)$$

for the autocorrelations orthogonal to  $\underline{\Omega}$ , and

$$R_{\times} = \pm e^{-t} \sin \omega t \quad (8)$$



for the correlations of orthogonal velocity components (at two different times). The plus/minus sign in (8) depends on whether  $\underline{\Omega}$  and the two orthogonal ordered-by-time velocity-component directions make a right-handed coordinate system.

The mean angular momentum (per unit mass) for a fluid particle follows as

$$\underline{h} = \langle \underline{x} \times \underline{u} \rangle = \int_0^t \langle \underline{u}(t') \times \underline{u}(t) \rangle dt',$$

thus, for  $h = |\underline{h}|$ , the mean angular momentum is

$$h = 2 \int_0^t R_{\times} d\tau \sim \frac{2\omega}{1+\omega^2} (t \gg 1) \quad (9)$$

and is of course parallel to  $\underline{\Omega}$ . Thus it is possible to determine the parameter  $\omega$  in terms of a well defined physical property, although evidently the magnitude of the angular momentum is bounded by unity. The reason for a bound on angular momentum is that for small  $\omega$  the spiralling rate (or angular velocity) of the trajectory about  $\underline{\Omega}$  is likewise small and therefore the angular momentum is small. For large  $\omega$ , the spiralling rate is large, but the excursions perpendicular to the axis are small (see below) which leads to small angular momentum. When there is a balance between the angular speed of fluid particles, and the propensity for them to migrate away from the axis, then we have maximal angular momentum.

It is also possible (Borgas, 1995) to explicitly relate  $\omega$  to joint Eulerian acceleration and velocity statistics:

$$\omega = \frac{1}{2} \langle \underline{u} \times \underline{a} \rangle \cdot \underline{\Omega} / |\underline{\Omega}|.$$

Thus  $\omega$  has a maximum magnitude of  $t_L \frac{\sigma_a}{\sigma_u}$  ( $\sigma_a^2 = \frac{1}{3} \langle \underline{a} \cdot \underline{a} \rangle$ ) which is of course large, but only possible if velocities are highly correlated with the large accelerations. It is more natural to assume that the correlations are weaker than the maximal and that  $\omega \sim 1$  is expected for the broken reflexion flows. Consideration of the Navier-Stokes equations for rotating homogeneous turbulence suggest that this is the case for moderate rotation rates ( $\sim t_L^{-1}$ ). These details may only be formally considered in the context of a higher-order stochastic models (for accelerations) and will not be considered any further here.

Because the system is linear the distribution of displacements remains Gaussian and therefore the transport of passive tracer of negligible molecular diffusivity (released at the origin say) has a mean distribution which is Gaussian too, with a characteristic 'width' in any direction determined by the mean-square fluid-particle displacements (dispersion) in that direction.

The dispersion for release at the origin according to the present model (in dimensionless form) is

$$\begin{aligned} \langle x_i x_j \rangle &= \int_0^t (t - \tau) (R_{ij}(\tau) + R_{ji}(\tau)) d\tau \\ &= 2D_{ij}(t); \end{aligned} \quad (10)$$

the two principle components of the dispersion are shown in figure 1 and are given as:

$$D_{\parallel} = t + e^{-t} - 1 \quad (11)$$

and

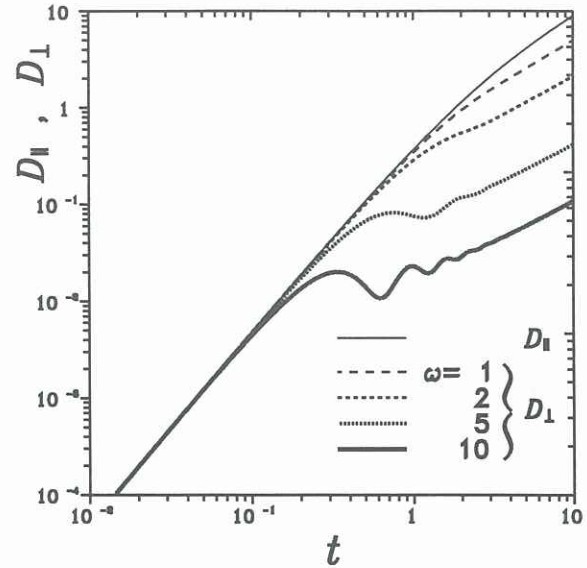


FIGURE 1. Turbulent dispersion components,  $D_{\parallel}$  and  $D_{\perp}$ , parallel and perpendicular to  $\underline{\Omega}$ , respectively. The dispersion becomes linear for sufficiently long times (eddy diffusion) but before doing so may show oscillations if  $\omega$  is large enough. The length-scale units are effectively  $\sigma_u t_L$ .

$$\begin{aligned} D_{\perp} &= \frac{t}{1+\omega^2} - \frac{2\omega e^{-t} \sin \omega t}{(1+\omega^2)^2} \\ &\quad + \frac{(\omega^2 - 1)(1 - e^{-t} \cos \omega t)}{(1+\omega^2)^2} \end{aligned} \quad (12)$$

The dispersion parallel to  $\underline{\Omega}$ ,  $D_{\parallel}$ , is identical to Taylor's result for isotropic turbulence, while the dispersion in any direction perpendicular to  $\underline{\Omega}$ ,  $D_{\perp}$ , is generally smaller than  $D_{\parallel}$ , so that the distribution becomes more and more 'needle-like' as  $\omega$  increases in size, and only when  $\omega$  vanishes is the distribution isotropic. At large times, the dispersion grows linearly with time as in a simple diffusion process, but the 'eddy diffusivities',  $D_{\parallel}$  and  $D_{\perp}$ , depend on direction, with transport orthogonal to  $\underline{\Omega}$  much reduced.

### PASSIVE TRACER FLUXES

Suppose now that we have a passive tracer ( $\theta$  say), of negligible molecular diffusivity, with a mean gradient of concentration,  $\underline{G}$  (say), in a direction orthogonal to  $\underline{\Omega}$ . Thus,  $\theta = \underline{G} \cdot \underline{x}$  at  $t = 0$ , but which is also the steady-state mean distribution of tracer. If the gradient is parallel to  $\underline{\Omega}$ , then the tracer is effectively well mixed in planes where the flow lacks reflexion symmetry, and the interesting oscillating components do not do anything special.

The maintenance of the constant gradient ( $g = |\underline{G}|$ ) requires fluxes of tracer which can be determined by

$$\begin{aligned} F_{\parallel} &= \langle \theta u_{\parallel} \rangle = g \int u_{\parallel} x_{0\parallel} P du_{\parallel} dx_{0\parallel} \\ &= g \int_0^t R_{\perp}(\tau) d\tau \\ &= g \frac{1 - e^{-t} \cos \omega t + \omega e^{-t} \sin \omega t}{1 + \omega^2} \end{aligned} \quad (13)$$

and

$$\begin{aligned} F_{\perp} &= \langle \theta u_{\perp} \rangle = g \int_0^t R_{\times}(\tau) d\tau \\ &= g \frac{\omega - \omega e^{-t} \cos \omega t - e^{-t} \sin \omega t}{1 + \omega^2}. \end{aligned} \quad (14)$$



Note that the 'parallel' and 'perpendicular' subscripts now refer to the direction of the concentration gradient,  $\underline{C}$ , which is orthogonal to  $\underline{\Omega}$ . Actually, (13) and (14) give the magnitude of the fluxes, the directions are of course down the gradient (in the direction  $-\underline{C}$ ) and transverse to the gradient (in the direction  $\underline{C} \times \underline{\Omega}$ ).

We find that there are two distinct turbulent fluxes, one down the gradient ( $F_{\parallel}$ ) and one orthogonal to the gradient ( $F_{\perp}$ ), which has been called a skew diffusion effect by Moffatt (1983). For small times there is little change to isotropic down-gradient transport and for large times the steady-state fluxes are

$$F_{\parallel} = \frac{g}{1 + \omega^2} \quad \text{and} \quad F_{\perp} = \frac{g\omega}{1 + \omega^2}.$$

For small  $\omega$  there is very little transverse flux and for very large  $\omega$  the transverse flux dominates over the down-gradient flux, however, both are small in this case. The maximal transverse flux occurs when  $\omega = 1$ , in which case the magnitudes of the two fluxes are equal, but only equal to one half of the flux for isotropic turbulence ( $\omega = 0$ ).

It is straightforward to understand the physical origin of the transverse fluxes. The flux at any point is determined by the average over all trajectories passing through that point. Consider the projection of two 'average' (but independent) trajectories in a plane orthogonal to  $\underline{\Omega}$ , one commencing above a measurement point  $M$  and one (by symmetry) commencing below it. Suppose that the right-handed sense of rotation of the trajectories dominates (because of lack of reflexion); of course, there is no net Eulerian velocity at  $M$  even with rotational asymmetry. The average transverse scalar transport at  $M$ , for a mean gradient that increases 'up the plane', follows because concentration is conserved along the trajectory, but is greater for the upper trajectory. Thus on average, trajectories commencing above the measurement point transport hotter fluid to the right (say) and trajectories below the measurement point transport colder fluid to the left. The net effect is a flux of warmer fluid to the right, i.e. a positive cross-gradient flux. Of course, there is always the simultaneous down-gradient flux.

## SUMMARY

We have briefly described some turbulent transport phenomena associated with flows with broken reflexion symmetry, where turbulent eddies have some preferred sense of rotation with respect to an axis of symmetry. Lagrangian fluid-particle trajectories spiral around the axis such that each particle has a net angular momentum which is a key measure of the flow character. These coherent spiral excursions lead oscillations in both the velocity auto- and cross-correlations, as well as reduced turbulent transport in planes orthogonal to the axis of symmetry. The magnitude of the changes, as well as the qualitative change to Taylor's (1921) isotropic-dispersion case, are quite surprising given the 'mildness' of the anisotropy considered.

A particularly noticeable feature is the prediction of 'skew' diffusion, where a transverse flux of material orthogonal to a concentration gradient of the material occurs simply by virtue of the preferred sense of rotation of the turbulent eddies. Despite this extra flux, the net flux from both the down-gradient and transverse fluxes, is of smaller magnitude than the down-gradient flux in isotropic turbulence for the same distribution of kinetic energy, energy dissipation rate, and material-concentration.

This work has broader implications for the stochastic modelling of Lagrangian trajectories in inhomogeneous (anisotropic) flows, because it is clear that other information, besides the Eulerian one-point distribution of velocity,

is required. At least for the present circumstances, this may mean incorporating angular-momentum statistics for fluid particles.

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