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STANDING WAVES ON CYLINDRICAL FLUID JETS

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ABSTRACT

An obstacle in the path of a cylindrical fluid creates standing waves upstream of the obstacle. The wavelengths and spatial damping coefficients of these waves is calculated as a function of jet velocity, for jets of different radii, using the linear dispersion relations. The effect of finite amplitudes on the wavelength is then determined using a perturbation method. The effect of finite amplitudes is found to be small for velocities at which standing waves are most clearly seen.

INTRODUCTION

A flat obstacle inserted in the path of a cylindrical fluid jet gives rise to a distinct standing wave pattern upstream of the obstacle. These waves can be easily observed on a low velocity jet issuing from a household tap[1], but, surprisingly, there is very little published work dealing with this phenomenon. The planar analogue of this phenomenon, however, is well known and has been the subject of several papers, many dating back to the last century[2]. Most of the analyses dealing with capillary waves on fluid cylinders deal with the instability that arises when the dimensionless wave number is less than unity. The linear inviscid problem was first solved by Rayleigh[3] in this context. Assuming axisymmetric waves of the form $e^{-i(kz-\omega t)}$ on an infinite jet of radius a and velocity v issuing in the z direction he obtained the dispersion relation

$$\omega = \sqrt{\frac{T}{\rho a^3}\frac{I_1(\alpha)}{I_0(\alpha)}\alpha(\alpha^2-1)}, \tag{1}$$

where $\alpha = ka$ is the dimensionless wave number and ρ and T are the fluid density and surface tension respectively. I_0 and I_1 are the modified Bessel functions of order 0 and 1. The linear viscous problem was also solved by Rayleigh[4], the resulting dispersion relation is,

$$2\alpha^2(y^2+\alpha^2)\frac{I_1'(\alpha)}{I_0(\alpha)}\left(1-\frac{2\alpha y}{y^2+\alpha^2}\frac{I_1'(y)I_1(\alpha)}{I_1(y)I_1'(\alpha)}\right)+(y^4-\alpha^4)-\frac{\Re^2}{\beta}\frac{I_1(\alpha)}{I_0(\alpha)}\alpha(1-\alpha^2)=0, \tag{2}$$

where, $y=\sqrt{\alpha^2+i\Re\omega}$, and $\Re$ is the Reynolds number, $\rho a v/\mu$, μ being the fluid viscosity. The dispersions (1) and (2) hold in a frame of reference moving with the jet.

In the 1970s the study of jet breakup was further catalysed by the development of ink-jet printing. During the past two decades several perturbative[5,6] and computational[7,8] calculations, which deal with the full nonlinear problem, have appeared in the literature. Most of these papers do not deal with stable waves; the category to which standing waves belong.

The few papers that deal with standing waves on cylindrical jets contain analyses based on equation (1)[9,10]. These investigations neglect viscous, finite amplitude and gravitational effects entirely. The main objective of those investigations was to carry out a measurement of the surface tension of the liquid forming the jet. However, the values of surface tension obtained may be in error because equation (1) does not account for the aforementioned effects. In this paper we investigate the effects of viscosity and finite amplitudes on the wavelength of standing waves, with the aim of devising a reliable and simple method of determining the dynamic surface tension of a fluid.

WAVELENGTHS AND DAMPING COEFFICIENTS OF STANDING WAVES

We have carried out calculations of the wavelength and damping coefficient of standing waves using equations (1) and (2). To obtain the wavelength and damping coefficient we set Re(ω) = α and Im(ω)=0 in equation (2) and solve the resulting transcendental equation. The equation is solved using the Newton-Raphson method. An initial guess for Re(α) is conveniently obtained from equation (1) and a guess for Im(α) from the expression for the spatial damping coefficient in the planar case.

The fluid considered is water (T = 0.073 N/m and $\mu = 0.001$ Pa-s). It is found that the wavelength of these waves is longest at low jet velocities and decreases dramatically as the velocity is increased (figure 1). This is in qualitative agreement with observations made on a ≈ 1 mm water jet issuing from a tap. The minimum jetting velocity for such a jet is ≈ 0.15 m/s. Below this velocity the flow is in the form of drops. From the plot it is also seen that at a given velocity the wavelength is greater for the thicker jet. It is difficult to see this variation in jets issuing from household taps as the radius cannot be varied independently of the flow rate. The situation is also complicated by gravity which is not included in the analysis. Note that the wavelength becomes independent of the jet radius at high velocities, corresponding to the planar limit, $ka \rightarrow \infty$.

Figure (2) is a plot of n, the number of wavelengths in which the amplitude decreases by a factor of e, as a function of the jet velocity. This quantity is essentially the inverse of the spatial damping coefficient. The most important feature is the rapid decrease of n at velocities well above the minimum jetting velocity implying that the waves are damped more rapidly at high velocities. This behaviour is qualitatively borne out by experiment. Note that n becomes independent of the jet radius at high jet velocities.

FINITE AMPLITUDE EFFECTS

In order to analyse finite amplitude effects we have determined the third order perturbative correction to the frequency assuming an inviscid incompressible fluid. Our treatment closely follows that of Wang[11], however our expressions have been recast in a form that permits a comparison with the planar case. The expression for the corrected frequency is found to be,

$$\omega = \omega_0 \left\{ 1 - \frac{\alpha^2}{16} C(\alpha) \left(\frac{\eta_0}{a} \right)^2 \right\}, \tag{3}$$

where ω_0 is the Rayleigh frequency (equation (1)), η_0, the amplitude of the disturbance and $C(\alpha)$ is a function of α listed in the appendix (see figure (3)). Note that in the limit $\alpha \rightarrow \infty$,corresponding to the planar case, $C(\alpha) \rightarrow 1$, and consequently equation (3) reduces to

$$\omega = \sqrt{\frac{Tk^3}{\rho}} \left(1 - \frac{k^2 \eta_0^2}{16} \right). \tag{4}$$

which is identical to the first two terms obtained by expanding Crapper's exact dispersion[12],

$$\omega = \sqrt{\frac{Tk^3}{\rho}} \left(1 + \frac{k^2 \eta_0^2}{4} \right)^{-\frac{1}{4}} \tag{5}$$

In figure (4) we show the effect of the correction term on the wavenumber of standing waves for jet velocities ranging from 0.05 to 0.7 m/s. It is evident that the lowest order nonlinear correction to the wavenumber is small at low velocities. The effect of higher order terms will be even smaller. It should be noted that the calculations cannot be extended all the way down to $\alpha = 1$ because of the singular behaviour of $C(\alpha)$. However, for all calculations presented here, the condition,

$$\frac{\alpha^2}{16} C(\alpha) \left(\frac{\eta_0}{a} \right)^2 << 1, \tag{4}$$

is satisfied.

In conclusion we would like to point out that most of the existing wave based methods for determining surface tension use results of a linear analysis[13]. The notable exception is Bohr's method[14]. However, his method depends crucially on the perfect ellipticity of the nozzle. It also requires high jet velocities for accuracy, which calls into question the assumption of laminarity of the flow. A method for measuring the dynamic surface tension of fluids based on standing waves will not suffer from these defects. We believe that the calculation presented in this paper puts the standing wave method on a firmer theoretical foundation.

APPENDIX

$C(\alpha)$ of equation (3) is given by,

$$C(\alpha) = C_1(\alpha) - C_2(\alpha), \tag{A1}$$

where,

$$C_1(\alpha) = 2A[f_1(2\alpha) - 2f_2(2\alpha)] + B[f_2(\alpha) - 2f_1(\alpha)] - 2 - f_3(\alpha), \tag{A2}$$

and,

$$C_2(\alpha) = \frac{(\alpha^2-1)\{\alpha^2[2A[f_1(\alpha)f_1(2\alpha)-1]+6f_1(\alpha)-f_2(\alpha)]-B[2\alpha f_1(\alpha)+\alpha^2]\}}{\alpha^2(\alpha^2-1)f_1(\alpha)} + \frac{f_1(\alpha)[-3\alpha^4+\alpha^2-6]}{\alpha^2(\alpha^2-1)f_1(\alpha)}, \tag{A3}$$

where,

$$A = \frac{\alpha(\alpha^2-1)[f_1(\alpha)^2-3]+f_1(\alpha)[\alpha^2+2]-\alpha f_1(\alpha)[f_1(\alpha)+f_2(\alpha)](1-4\alpha^2)}{\alpha f_1(\alpha)[1-4\alpha^2]+2\alpha(\alpha^2-1)f_1(2\alpha)}, \tag{A4}$$

and,

$$B = \frac{\alpha(\alpha^2-1)[f_1(\alpha)^2-3]+f_1(\alpha)[\alpha^2+2]+2\alpha(\alpha^2-1)f_1(2\alpha)[f_1(\alpha)+f_2(\alpha)]}{\alpha f_1(\alpha)[1-4\alpha^2]+2\alpha(\alpha^2-1)f_1(2\alpha)}, \tag{A5}$$

with $f_1(\alpha) = I_0(\alpha)/I_1(\alpha)$ and $f_2(\alpha) = I_1'(\alpha)/I_1(\alpha)$. and $f_3(\alpha) = I_1''(\alpha)/I_1(\alpha)$.

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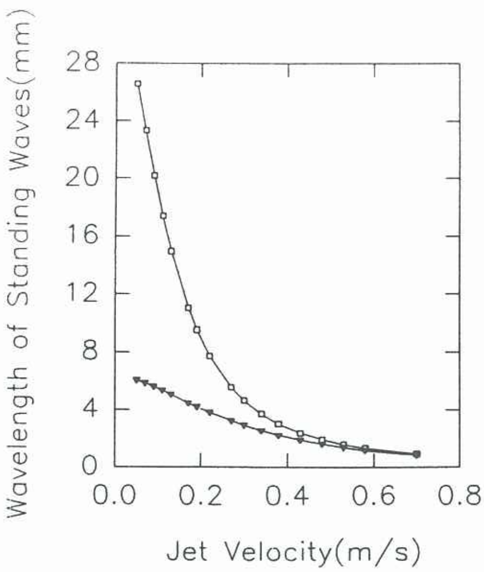

Figure 1

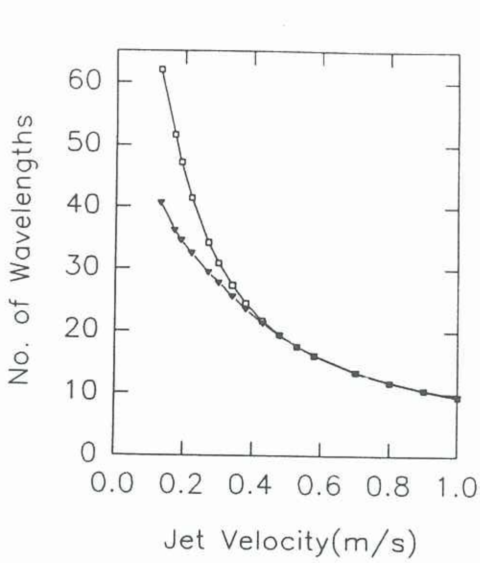

Figure 2

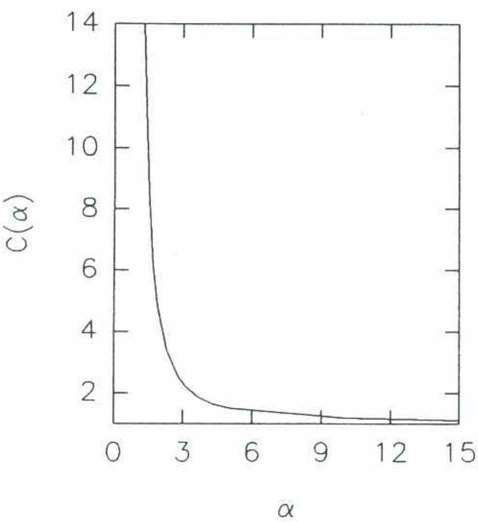

Figure 3

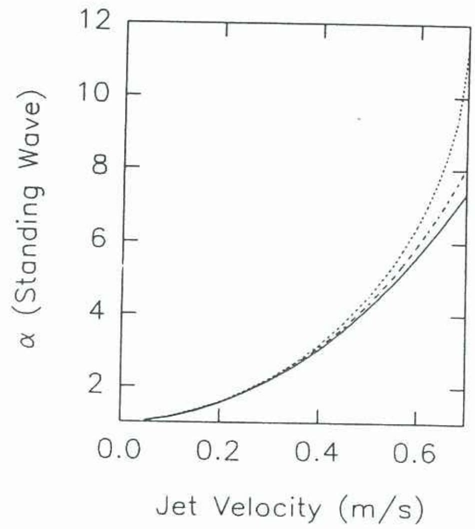

Figure 4

Figure 1: Wavelength of standing wave as a function of jet velocity. The triangle denotes the 1 mm radius jet and the unfilled square denotes the 5 mm jet.
Figure 2: Number of wavelengths in which the wave amplitude falls by a factor of e plotted as a function of jet velocity. The symbols have the same meaning as in figure 1.
Figure 3: $C(\alpha)$ as a function of α.
Figure 4: Standing wave number as a function of jet velocity for a water jet of radius 1 mm. The dotted line represents $\eta_0/a = 0.15$, the dashed line represents $\eta_0/a = 0.1$ and the solid line is the linear solution.