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A NON-LINEAR EVOLUTION EQUATION FOR WIND DRIVEN, BREAKING WAVES AND RESULTING SOLITON WAVE GROUPS

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Abstract

The third order dynamical model (cubic non-linear Schrodinger equation) describing the evolution of a narrow banded surface gravity wave distribution is extended to the case where wind input, breaking dissipation, and turbulent diffusion are present. The modeling of real ocean effects is self-consistent and is based on the Kitaigorodskii-Toba energy spectrum and Kahma' law of wind input. One consequence of this new evolution equation is described, the existence of soliton wave groups whose steepness envelope has the familiar sech form, and is dependent only on the wind speed. The number of waves within these groups is in good agreement with ocean observations of energetic wave groups.

Notation

k, ω, c	wave number, frequency, velocity
$x, z \rightarrow kx, kz$	dimensionless horizontal coordinates
$t \rightarrow \omega t$	dimensionless time
$X = x - 1/2\, t$	horizontal moving coordinate
$A \rightarrow kA$	dimensionless wave amplitude
$\phi \rightarrow k^2\phi/\omega$	dimensionless wave potential
p, ρ, g	pressure on surface, density, gravity
U, u_*	wind speed, friction velocity
β	wind-wave growth rate
δ	dissipation rate coefficient
D	diffusion damping coefficient
Φ	wave frequency spectrum
Ω	dimensionless frequency

$\mu = U/c$, and $\mu_* = u_*/c$.

Introduction

The observed self-similar sea spectra is narrow or quasi-monochromatic (the half-width is 0.15 of the peak frequency), and the wave slopes are small, $O(10^{-1})$. These facts alone suggest that the energetic sea might be treated as a weakly non-linear, narrow-banded, dynamical system. The sea spectra is known to be controlled by the wind and by the balance between wind energy input and dissipation. Here we undertake the modelling of this locally non-conservative system by an extension of the nonlinear evolution equation of the cubic nonlinear Schrodinger type, first derived for weakly nonlinear dissipative systems by Benny and Newell (1967), and first applied to conservative water wave dynamics by Zakharov (1968).

In accord with field measurements, we take the wind input to be linear and the breaking dissipation to be cubic in the wave amplitude, and each term is much smaller (by a factor $O(10^{-2})$ than the inertial terms in the Zakharov formulation. Nevertheless, over sufficiently long times, $O(10^4$ wave periods), the non-conservative terms dominate the behavior of the system. Then, entirely new phenomena appear in comparison to the conservative system. These are: (i) the appearance of propagating wave groups of permanent form (solitons) whose characteristics are entirely determined by the wind speed; and (ii) the ultimate appearance of 3 wave resonant side-band systems undergoing modulated interactions about an attractor dependent on the wind speed. For lack of space, we discuss only the first of these new phenomena here. We show that there is very good agreement between the predicted and observed wave groupings.

The shape of the energetic wave energy spectrum is crucial for modeling, and we rely on the Kitaigorodskii-Toba (K-T) spectrum, Kitaigorodskii (1962) and Toba (1973):

$$\Phi(\omega) = \alpha_{KT}\, g\, u_*\, \omega^{-4} \qquad \alpha_{KT} = 0.02 \qquad (1)$$

which is widely accepted to model the energetic wind-wave spectra for $\omega_p < \omega < 3\omega_p$. For much shorter gravity waves, $\Phi \sim \omega^{-5}$, as first suggested by Phillips (1958). The choice of the wind input model cannot be made independent of the choice of spectral laws, as we discuss subsequently, and the dissipation must in turn match the wind input. Rather remarkably, then, the modeling of all real effects eventually hinges on (1).

The Evolution Equation

For a weakly nonlinear quasi-monochromatic wave with a central wavenumber k and frequency ω, the evolution of its dynamics for the non-dimensionalized complex wave amplitude, A(x,z,t), is described by the following equation of generalized Ginzburg-Landau type:

$$\begin{aligned} & A_t + \frac{1}{2} A_x + i \frac{1}{8}(A_{xx} - 2A_{zz}) + i \frac{1}{2} A|A|^2 \\ & \quad + i\, A(\phi_{10})_x \\ & \quad - \frac{1}{16}(A_{xxx} - 6A_{xzz}) - \frac{1}{4}(A^2 A_x^* - 6|A|^2 A_x) \\ & \quad = R(A) \end{aligned} \qquad (2)$$

where the first line represents dominant inertial effects, and when set equal to zero corresponds to the cubic-Schrodinger formulation of Benny and Newell (1967) and Zakharov (1968); the second line represents the effect of an underlying current introduced by Davey and Stewartson (1974); the third line represents quartic inertial terms introduced by Roskes (1977) and Dysthe (1979). The generalized terms on the fourth line represents the effects of wind input (pumping) and dissipation (damping), which are introduced below. All of the conservative terms on the left-hand-side of (2) may be derived by a multiple scale analysis of the surface gravity wave problem.

The behavior of conservative systems governed by (2) in the absence of R(A) has been studied extensively. Soliton wave group solutions of the cubic conservative system exist and were shown experimentally by Yuen and Lake (1975). Other wave groups have been shown to self-modulate, leading to a long period recurrence. The fourth order terms introduce asymmetries in the wave group shapes (steepening of the leading face of the groups) and tend to unbalance the distribution of energy in the resonant side-bands, in favor of the low frequency band (downshifting). The existence of resonant side-band systems, first explored by Benjamin and Feir (1967), arises from the conservative form of (2), whereby they may be treated. Despite the great success of the conservative weakly nonlinear formulation, it fails in applications to the wind-driven sea; the reason for this is that the non-conservative terms, represented by R(A), dominate the eventual behavior of wave groups, despite the smallness of the R(A) terms. This is very fortunate as it results in the prediction here of nonlinear waves dependent on the wind speed and wave age, just as observed in practice.

Previous attempts to study the effects of wind input and/or breaking dissipation on (2) have concentrated on other phenomena than are studied here. These include the introduction of nonlinear breaking dissipation including an onset wave slope, to show downshifting under the influence of breaking, Trulsen and Dysthe (1990), and an extension to include the effects of wind input, Trulsen and Dysthe (1992). In the latter work, a suppression of modulational instability was found at higher wind speeds. Down-shifting was also studied by Hara and Mei (1991) under the influence of wind input and of linear damping. In all of these works the wind was modeled according to the formulation of Plant (1982), which is intended for input to shorter waves, and is not applicable to the energetic spectrum; nor does the modeling in these past works result in the generation of spectral shapes as have been measured in the ocean.

Wind Input to Waves

The wind induced pressure on the sea surface, p_w, can be safely assumed proportional to the wave slope, including terms in and out of phase with the wave elevation, Phillips (1963):

$$p_w/\rho = -i\, c^2 (\alpha + i\, \beta)\, \partial\eta/\partial x \qquad (3)$$

It is generally believed now that the β part of p_w is directly responsible for the wind energy input to waves and it has been measured both in small wavetanks and in the open sea; the α term has a very small effect on the wave phase and will hereafter be neglected. The latest ocean measurements, Hasselmann et al (1991), suggests the following law for the nondimensional growth rate of individual waves:

$$\beta_I = .25\, \rho_a/\rho\, (\mu - 1) \qquad 1 < \mu < 3 \qquad (4)$$

where $\mu = U/c$. This growth rate is particularly relevant for the larger energetic waves. It is also consistent with that part of the physics implied by Miles (1957) theory of wave generation by wind which requires the input to disappear when the wind speed and wave speed are equal ($\mu = 1$).

It may also be asked what is the effective growth rate, β, for the entire energetic spectrum. It has been shown, Kahma (1981), that in the case of K-T spectral shape, the effective growth rate may be calculated from a knowledge of the spectral constants and of the wind friction coefficient. He found:

$$\beta = K\, \mu_*, \qquad K = 6.39 \times 10^{-3} \qquad (5)$$

where the inverse wave age $\mu_* = u_*/c_p$, and c_p refers to values at the spectral peak; it needs to be noted that for small, high frequency waves, ($\mu_* > 0.1$), a quadratic law, $\beta \sim \mu_*^2$ has been observed, Plant (1982), consistent with the Phillips (1958) short wave equilibrium spectrum.

In our own modeling we favor (5), since it leads to a self-consistent model, provided that the K-T spectral shape applies in the energetic wave range, as we assume here.

The wind pressure (3) when introduced into the mathematical derivation of (2) results in a linear term on the r-h-s, $\beta/2$ A.

Dissipation of Wind Waves

The turbulent dissipation which accompanies the strong mixing in the breaking wave is responsible for irreversible energy losses in energetic wind wave systems. A detailed theory of breaking losses is not available, nor is an adequate understanding of the cause and mechanism accompanying energetic breaking, see Tulin and Li (1992). In view of the disparity of energy in wind and the largest waves, it seems likely that the breaking mechanism and its details do not depend directly on the wind, but can depend only on the wave steepness, $|A|$. It is necessary in an equilibrium sea that wind input and breaking dissipation match when integrated over the entire spectrum. If the dissipation rate ($E^{-1}dE/dt$) is assumed $\sim \delta |A|^{2(n-1)}$, then this matching allows a determination of both the exponent, n, and of dissipation rate constant, δ. In the case of a Stokes wave, then, assuming matching of breaking and wind input and using (5):

$$\delta |A|^{2(n-1)} = \beta = K \mu_* \tag{6}$$

The steepness of the waves in the K-T spectra is simply related to μ_* according to a law given explicit form by Toba (1972):

$$|A|^2 = \pi^3 B^2 \mu_* \qquad (B = 0.062) \tag{7}$$

As a result,

$$n = 2; \qquad \delta = K/\pi^3 B^2 = 0.0536. \tag{8}$$

It may be noted that a scaling of the breaking dissipation with $|A|^{2(n-1)}$ has been proposed before by various authors: Hasselmann (1974), n = 1; Phillips (1985), n = 3; Plant (1986), n = 2; Donelan and Pierson (1987). We have found above n = 2 necessary for self-consistency with both the K-T spectrum, and Kahma's law of wind input, which follows from the K-T spectrum.

The evolution equation which now results from previous considerations is (we omit the fourth order terms, as non-essential):

$$A_t + \frac{1}{2} A_x + i\frac{1}{8}(A_{xx} - 2A_{zz}) + i\frac{1}{2} A|A|^2 = \frac{1}{2}\beta A + D A_{xx} - \frac{1}{2}\delta |A|^2 A \tag{9}$$

It is also useful to express this in a coordinate system moving with the group velocity (1/2, in non-dimensional terms),

$$A_t + i\frac{1}{8}(A_{XX} - 2A_{zz}) + i\frac{1}{2} A|A|^2 = \frac{1}{2}\beta A + D A_{XX} - \frac{1}{2}\delta |A|^2 A \tag{10}$$

where we have omitted the effects of underlying current, as we shall not deal with them in the following.

Soliton Wave Groups

It was first observed at sea by Donelan, Longuet-Higgins, and Turner (1972) that clearly discernable energetic wave groups are present, and this was later confirmed by Thorpe and Humphries (1980), Su (1986), and Holthuisjen and Herbers (1986). Controversy exists as to whether these are the consequence of statistical properties of a stochastic linear wave system, or whether they are a product of non-linear wave dynamics; Su (1986) conducted a detailed analysis of wave groups measured in the Gulf of Mexico and compared with nonlinear resonant wave group characteristics, and strongly argued in favor of nonlinear wave group formation. However, a theory of wave group formation based on wind generation has not heretofore been given.

The model evolution equation (10) has a soliton wave group solution given by

$$A = a\,[\mathrm{sech}(bX)]^{1+i\theta}\, e^{-i\Omega t} \tag{11}$$

where b is very closely approximated by $b = \sqrt{2}\,a$, so that

$$|A| = a\,\mathrm{sech}(\sqrt{2}aX) \tag{12}$$

and

$$a^2 = 3/2\,(\beta/\delta) = (3/2\,\pi^3 B^2)\mu_* = 3/2\,|A|^2_{Toba} \tag{13}$$

The number of waves within a spatial envelope, N_s, thus depends only on the steepness of the waves within the envelope and this allows a direct comparison with ocean data (using Su's (1986) data analysis). We find, taking $a_{max} = .2$:

	Stokes	Su (measured)
a_{mean}	.14	.14
N_s	7/2	3

Note that the amplitude of the soliton and therefore the size of the wave group is, (13), entirely determined by the wave age, μ_*^{-1}, without any disposable parameters. While the good agreement with Su's data is highly suggestive, the soliton (12) exists in the absence of background waves, unlike those wave groups observed at sea.

Summary and Conclusions.

(1). The effect of wind, breaking, and turbulent diffusion on the generation and propagation of narrow banded surface wave groups has been modeled, and an appropriate evolution equation derived as a modification of the well-known conservative evolution equation of the cubic-nonlinear Schrodinger type.

(2). The modeling of real effects is self-consistent and hinges on the validity of the Kitaigorodskii-Toba spectral law for energetic waves. This law implies the Kahma linear law for wind energy input, and the later implies an energy dissipation rate depending on the fourth power of the wave steepness. A self-consistent theory results in which all the constants are known from ocean data.

(3). A general development suggests the existence of turbulent diffusion of the wave groups in the presence of breaking, but the associated diffusion coefficient is not known at this time. Although not discussed here, it is worth noting that, if large enough, diffusion results in a quantitative change in the behavior of the wind driven dynamical system.

(4). A soliton wave group of the classical sech form exists as a solution of the wind-driven evolution equation. In this solution, the width of the soliton (or number of waves in the group) depends only on the steepness amplitude of the wave group, which in turn depends only on the wave age (no disposable constant). Good agreement between predicted and observed wave group size is found.

(5). In a further development of the work reported here, the long term evolution of narrow banded distributions has been studied using this wind-driven evolution equation and modulation has found about a wind dependent attractor.

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