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A FIRST COURSE IN OPTIMUM DESIGN OF YACHT SAILS

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ABSTRACT

The optimum sail geometry is analytically obtained for the case of maximizing the thrust under equality and inequality constraints on the lift and the heeling moment. A single mainsail is assumed to be set close-hauled in uniform wind and upright on the flat sea surface. Governing parameters are the mast height, and the gap between the sail foot and the sea surface. The lifting line theory is applied to analyze aerodynamic forces acting on a sail. The design method consists of the variational principle and the feasibility study. Almost triangular sails are found to be optimum.

INTRODUCTION

Analyses of sail aerodynamics have been conducted by Tanner (1967) and Milgram (1968) by using the lifting line theory. They concluded that sail designers should trade off drag reduction against the heeling moment constraint.

Sparenberg and Wiersma (1976) have treated this trade-off problem by introducing Munk's variational principle (Munk, 1919) into analysis of sails. They presented asymptotic and numerical solutions for the optimum circulation that maximize the thrust with a given side force and a given heeling moment. Wood and Tan (1978) also computed the optimum circulation about sails having the maximum thrust and a given heeling moment through trial-and-error procedure. Both design methods, however, lack the feasibility study, and present the optimum circulation only. Hence it is an open question if all their results could be realized even within the framework of their theories.

This study presents the optimum sail geometry through the optimization and the feasibility study.

DESIGN GOAL

The most important performance of a yacht is its ability of sailing to windward. This performance can be optimized by maximizing the thrust.

In steady-state sailing close to wind, aerodynamic and hydrodynamic forces are in equilibrium as is shown in Fig. 2. Considering hydrodynamic characteristics of hulls, the conventional wing theory implies that the boat drag, hence the thrust, can be approximated as a quadratic function of the side force. Therefore the side force, hence the lift, shall be maximum to maximize the thrust.

The bending moment at the mast root is closely related to the strength of the mast, and corresponds to the heeling moment of yachts. The heeling moment and the righting moment are in equilibrium. The righting moment has its uppermost value, because this moment is produced by weights of helmsmen and/or weight and buoyancy of the boat. Therefore, either the mast strength or the righting moment constrains the heeling moment.

To summarize, our design goal is the maximization of the aerodynamic thrust under the equality constraint on the lift and the inequality constraint on the heeling moment.

MODEL AND ASSUMPTIONS

We assume that the air flow is steady and inviscid, and that the sea surface is flat.

We treat upright single mainsails that are set flat. In this first course study, we neglect the elasticity of fabric sails.

Yacht sails are equivalent to wings of high aspect ratio with mirror images reflected on the sea surface. It is appropriate in the first approximation that we treat inviscid flow past them by using the lifting line theory: approximating wings as horse-shoe vortex distributions.

We neglect the presence of hulls. It is noted that the presence of hulls reduces the effective gap between the sail foot and the sea surface. Hence the negligence of hulls may bring some safety margin to our design.

BASIC EQUATIONS

In steady-state sailing close to wind, aerodynamic and hydrodynamic forces are in equilibrium (Fig.1);

$$T = L\sin\phi - D\cos\phi \cong L\phi - D\,, \tag{1}$$

$$S = L\cos\phi + D\sin\phi \cong L\,, \tag{2}$$

where T, L, D, ϕ, and S denote the thrust, the lift, the induced drag, the angle of sailing course to the relative wind, and the side force, respectively.

Aerodynamic forces and the heeling moment are given on the basis of the lifting line theory;

$$L = \rho V_R^2 h^2 \int_\delta^1 \gamma(z)dz\,, \tag{3}$$

$$D = \rho V_R^2 h^2 \int_\delta^1 \gamma(z)\alpha_i(z)dz\,, \tag{4}$$

$$M = \rho V_R^2 h^3 \int_\delta^1 \{\gamma(z)\cos\phi + \gamma(z)\alpha_i(z)\sin\phi\}zdz$$
$$\cong \rho V_R^2 h^3 \int_\delta^1 \gamma(z)zdz\,, \tag{5}$$

where M, ρ, V_R, h, γ, and α_i denote the heeling moment, the air density, the relative wind velocity, the mast height, the dimensionless circulation, the induced dangle of attack, and the dimensionless gap between the sail foot and the sea surface, respectively. The variable z is made dimensionless by using h.

The lifting line integral equation with a mirror image of the vortex system is given by

$$\alpha_i(z) = -\frac{1}{4\pi}\int_{\delta}^{1} \frac{d\gamma}{d\zeta}\left\{\frac{1}{\zeta - z} + \frac{1}{\zeta + z}\right\} d\zeta, \tag{6}$$

where δ denotes the dimensionless gap between the sail foot and the sea surface. By using a change of variables, this equation becomes akin to the conventional lifting line integral equation. Once we invert the transformed equation by using the well-known formula, then through reversion of variables we have

$$\frac{d\gamma}{dz} = \frac{4}{\pi\sqrt{(z^2-\delta^2)(1-z^2)}}\int_{\delta}^{1}\sqrt{(\zeta^2-\delta^2)(1-\zeta^2)}\,\alpha_i(\zeta)\left\{\frac{1}{\zeta - z} + \frac{1}{\zeta + z}\right\} d\zeta$$

$$+\frac{C}{\sqrt{(z^2-\delta^2)(1-z^2)}}, \tag{7}$$

where C is an arbitrary constant to be determined so that $\gamma(z)$ satisfies the boundary conditions $\gamma(\delta)=\gamma(1)=0$.

The theorem of Kutta and Zhukovski gives

$$\gamma(z) = \frac{1}{2} C_{l_\alpha}\{\alpha - \alpha_i(z)\} c(z), \tag{8}$$

where $C_{l\alpha}$, α and $c(z)$ denote the local lift curve slope assumed to be 2π, the geometrical angle of attack and the dimensionless chord, respectively. Solving Eq.(8) with respect to $c(z)$, we have

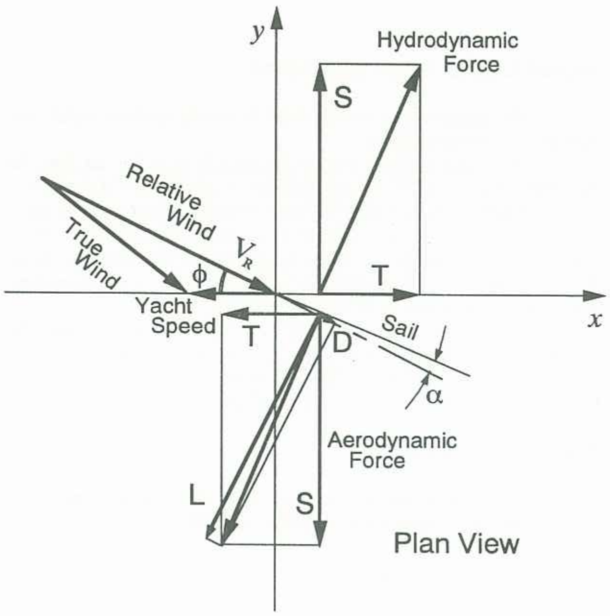
$$c(z) = \frac{\gamma(z)}{\pi\{\alpha - \alpha_i(z)\}}. \tag{9}$$

VARIATIONAL PRINCIPLE

Our design concept can be expressed by using variational principle, that is maximization of the functional

$$\int_{\delta}^{1} \gamma(z)\{\phi - \alpha_i(z)\} dz, \tag{10}$$

that corresponds to the sail thrust, accompanied by Eq.(6) and the constraints:



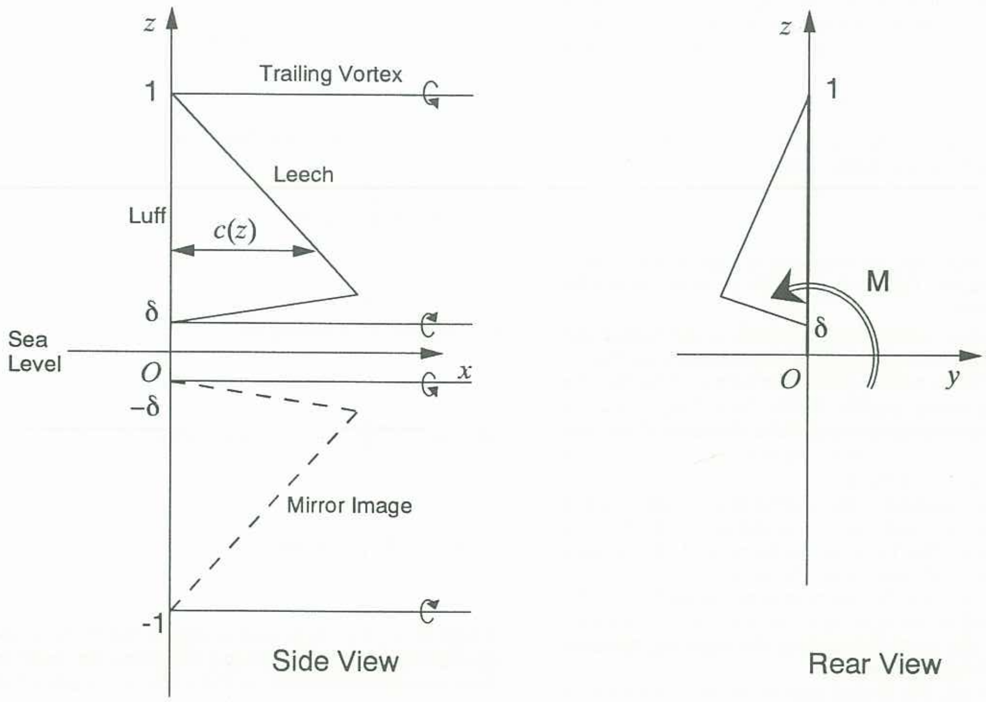


Figure 1. Nomenclature

$$l_{max} = \int_\delta^1 \gamma(z)dz \quad , \text{ and } \quad m_{max} \geq \int_\delta^1 \gamma(z)z\, dz \quad , \tag{11}$$

where l_{max} and m_{max} denote upper limits of definite integrals. Let us use values of the elliptic loading, $\gamma_R\sqrt{1-z^2}$, with the mast height, h_e, as a reference, and we have

$$l_{max} = \frac{\pi\gamma_R}{4\mu^2}, \text{ and } \quad m_{max} = \frac{\gamma_R}{3\mu^3}, \tag{12}$$

where μ is the mast height ratio defined by h/h_e.

Introducing Lagrange multipliers, λ_l, λ_m and λ_α, and a slack variable, ξ, we have the extended functional:

$$\int_\delta^1 \gamma(z)\left\{\phi - \alpha_i(z)\right\}dz$$
$$+ \lambda_l\left\{l_{max} - \int_\delta^1 \gamma(z)dz\right\} + \lambda_m\left\{m_{max} - \int_\delta^1 \gamma(z)zdz - \xi^2\right\}$$
$$+ \int_\delta^1 \lambda_\alpha\left[\alpha_i(z) + \frac{1}{4\pi}\int_\delta^1 \frac{d\gamma}{d\zeta}\left(\frac{1}{\zeta - z} + \frac{1}{\zeta + z}\right)d\zeta\right]dz \quad . \tag{13}$$

Take the first variation of the functional above with respect to λ_l, λ_m, λ_α, ξ, α_i and γ, and we have six stationary conditions. From the first three, we regain constraints on the lift and the heeling moment, and Eq.(6). The next condition is $\lambda_m\xi=0$: $\lambda_m=0$ or $\xi=0$. Hence there are two possible cases. The stationary condition last but one yields the relation: $\lambda_\alpha=\gamma(z)$. Using this relation, the last stationary condition yields the boundary condition required for the optimum solutions:

$$\alpha_i(z) = (\phi - \lambda_l - \lambda_m z)/2. \tag{14}$$

Substituting Eq.(14) into Eq.(7) and using boundary conditions on $\gamma(z)$, we have

$$\frac{d\gamma}{dz} = \frac{2}{\pi\sqrt{(z^2-\delta^2)(1-z^2)}}$$
$$\times \left[\left(\frac{\mathbf{E}}{\mathbf{K}} - z^2\right)\left\{\pi(\phi - \lambda_l) - 2\mathbf{E}\lambda_m\right\} + \lambda_l\left(\frac{\mathbf{E}^2}{\mathbf{K}} - \delta^2\mathbf{K}\right)\right.$$
$$\left. - 2\lambda_m(z^2-\delta^2)(1-z^2)\int_0^1 \frac{1}{1-z^2-k^2u^2}\sqrt{\frac{1-k^2u^2}{1-u^2}}du\right], \tag{15}$$

where $\mathbf{K}$, $\mathbf{E}$ and k denote the complete elliptic integrals of the first kind and the second, and the modulus, respectively;

$$\mathbf{K} = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \quad \mathbf{E} = \int_0^1 \sqrt{\frac{1-k^2u^2}{1-u^2}}du, \text{ and } k = \sqrt{1-\delta^2}.$$

Equation (16) yields

$$\int_\delta^1 \gamma(z)dz = \frac{\pi}{2}\left(1+\delta^2 - 2\frac{\mathbf{E}}{\mathbf{K}}\right)(\phi - \lambda_l)$$
$$- \frac{2}{3}\left\{(1+\delta^2)\mathbf{E} - \delta^2\frac{\mathbf{K}}{2} - \frac{3\mathbf{E}^2}{2\mathbf{K}}\right\}\lambda_m, \tag{16}$$

and

$$\int_\delta^1 \gamma(z)z\, dz = \frac{2}{3}\left\{(1+\delta^2)\mathbf{E} - \delta^2\frac{\mathbf{K}}{2} - \frac{3\mathbf{E}^2}{2\mathbf{K}}\right\}(\phi - \lambda_l)$$
$$- \frac{\mathbf{E}}{\pi}\left\{(1+\delta^2)\mathbf{E} - \delta^2\mathbf{K} - \frac{\mathbf{E}^2}{\mathbf{K}}\right\}\lambda_m. \tag{17}$$

Case 1: $\xi=0$

When the slack variable ξ is zero, Eq.(13) implies that the moment is equal to the given value. Equating Eqs.(16) and (17) respectively to their maximum values, Eq.(12), we have a set of simultaneous equations with respect to λ_l and λ_m Solving these equations, we have λ_l and λ_m as functions of μ and δ.

Case 2: $\xi\neq0$

When ξ is not zero, λ_m is zero. Solving Eqs.(11), (12) and (16), we have λ_l as a function of μ and δ. As λ_m is zero, the moment is not more than the maximum value. This condition is written in the form;

$$\mu < \mu_m , \tag{18}$$

where μ_m is a parameter dependent on δ only.

The close examination reveals that Case 1 has greater thrust than Case 2.

FEASIBILITY STUDY AND OPTIMIZATION OF MAST HEIGHT AND GAP

Feasible solutions must satisfy the constraint on the chord: $c(z)\geq0$. Equations (6) and (9) imply that this feasibility is assured by positive $\gamma(z)$. Some consideration leads to the conclusion: $\gamma(z)$ is positive, if

$$\mu \leq \mu_c, \tag{19}$$

where μ_c is a function of δ, and no less than μ_m.

Let μ_{opt} denote μ in the segment of $[\mu_m, \ \mu_c]$ that maximizes the sail thrust, and after considerable calculation we have

$$\mu_{opt} = \frac{4}{3}\,\frac{1+\delta^2 - \frac{2\mathbf{E}}{\mathbf{K}}}{(1+\delta^2)\mathbf{E} - \delta^2\mathbf{K} - \frac{\mathbf{E}^2}{\mathbf{K}}} . \tag{20}$$

The optimum ratio of mast height, μ_{opt}, is no more than 4/3. If and only if δ is zero, μ_{opt} coincides with μ_c at μ=4/3. Therefore, the optimum gap shall be zero.

Figure 2 shows the relation among μ_m, μ_{opt} and μ_c against δ.

NUMERICAL RESULTS AND DISCUSSION

Figure 3 shows the optimum circulation distributions for δ=0, 0.05, 0.1 and 0.15 together with the elliptic loading as a reference. Enclosed areas correspond to total side forces and all are the same. Most of the necessary lift is produced in the vicinity of the sail foot so that the heeling moment would not exceed the maximum value. Since the mast height becomes smaller for larger δ, even optimum sails of large δ have rather smaller aspect ratios.

Figure 4 shows the optimum sail geometry when the effective angle of attack at the sail foot, $\alpha-\alpha_i(\delta)$, is $2\gamma_R\pi$. It is noted that the practical triangular sail is fairly close to the optimum geometry of zero gap. We have the geometry of this ultimately optimum sail with no gap given by elementary functions as

$$c(z) = \frac{\sqrt{1-z^2} + \frac{z^2}{2} ln\left|\frac{1-\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\right|}{\pi\left\{\frac{32\alpha}{27\gamma_R} - \frac{1}{2} + \frac{\pi z}{4}\right\}}. \quad (20)$$

Let us introduce the thrust ratio:

$$\frac{T}{T_e} = \frac{(L/D_e)\phi - D/D_e}{(L/D_e)\phi - 1}, \quad (21)$$

where quantities with subscript "e" denote those of the elliptic loading.

Figure 5 shows the numerical results of thrust and drag ratios against gap δ, in case L/D_e and ϕ are equal to $30/\pi$ ($\cong 10$) and $\pi/5$ (36 deg.). It is quite evident from this figure that the presence of the gap between the sail foot and the sea surface enhances the drag and reduces the thrust considerably.

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REFERENCES

MILGRAM, J H (1968) The Analytical Design of Yacht Sails. Trans Soc Nav Archit Mar Eng, 76, 118-159

MUNK, M (1919) Isoperimetrische Aufgaben aus der Theorie des Fluges. PhD Thesis, University of Goettingen

SPARENBERG, J A and WIERSMA, A K (1976) On the Maximum Thrust of Sails by Sailing Close to Wind. J Ship Res, 20, 98-106

TANNER, T (1967) The Application of Lifting Line Theory to an Upright Bermudan Mainsail. J R Aeronaut Soc, 71, 553-558

WOOD, C J and TAN, S H (1978) Towards an Optimum Yacht Sail. J Fluid Mech, 85, 459-477

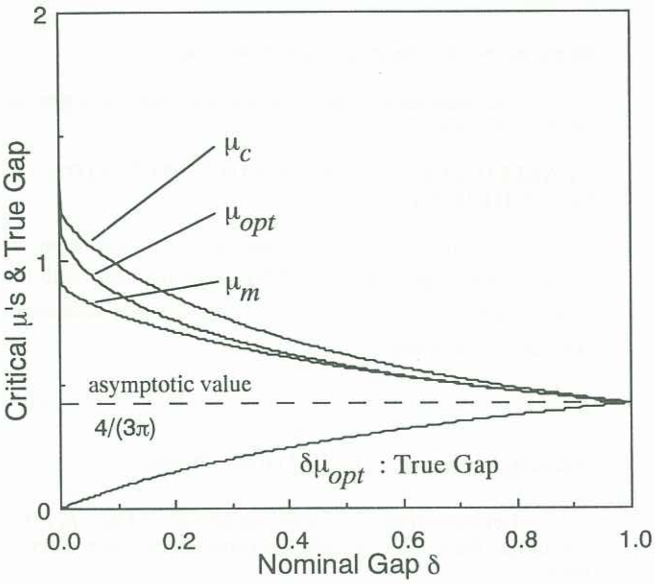

Figure 2. Relation among Critical Mast Height Ratios against Nominal Gap

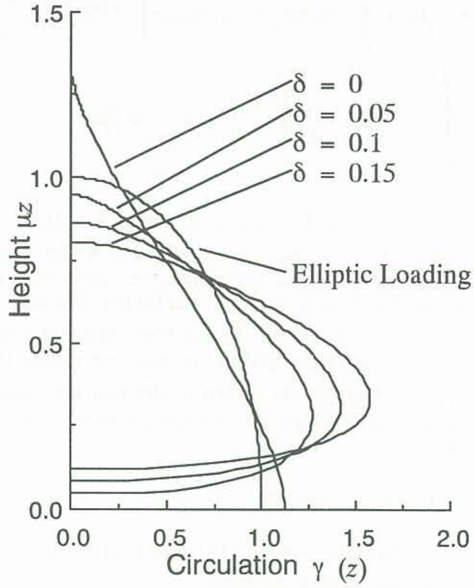

Figure 3. Optimum Circulation Distributions

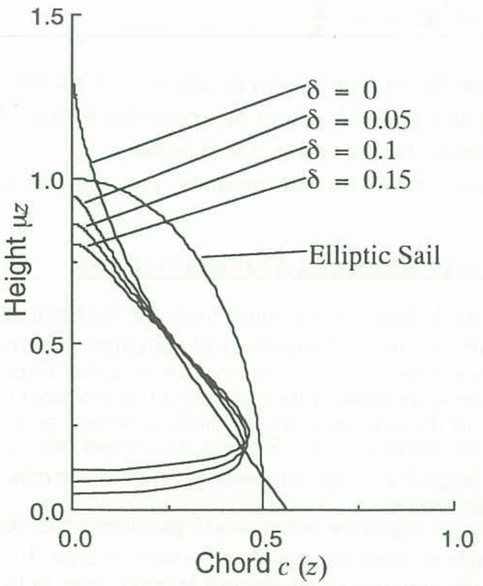

Figure 4. Optimum Sail Geometry

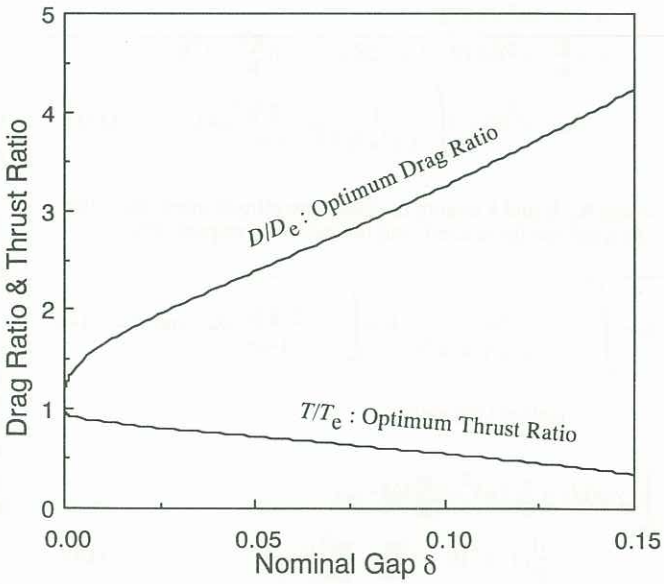

Figure 5. Thrust and Drag Ratios