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SOLUTE TRANSPORT IN HILLSIDE SEEPAGE

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ABSTRACT: The accurate prediction of solute transport in long, shallow hillslopes is fraught with difficulties, when numerical schemes are used. Most of the flow field will be uniform, however, when the length to depth ratio is greater than 20:1. This facilitates the use of series solutions for the stream function and the velocity field to transform the flow domain to the unit square. The transport equation is then solved using a series expansion, assuming a uniform flow field, and transformed back to the original flow domain. This approach avoids the computation and numerical pitfalls associated with solving the transport equation in the original flow domain. Solutions in both flow domains are presented and the results discussed.

1 INTRODUCTION

An understanding of hillslope seepage is important in managing and conserving the quality of surrounding landscapes and water resources. Increases in groundwater recharge can result from the clearing of large vegetation and the introduction of irrigation schemes, thereby producing a higher water table elevation. Solute transport occurs when the elevated water table mobilises salts from reservoirs, located in the newly saturated zone. The mobilised salts are then advected and dispersed in the aquifer, and may eventually be transported to the surface at the seepage face. Efficient land useage and conservation requires quantitative knowledge of the diffusion and advection processes of the solute through the saturated zone.

Two equations govern solute transport in porous media, namely the flow equation and the mass transport equation. The flow equation determines the seepage velocities, while solute concentrations are obtained using the mass transport equation. The velocity field can be determined continuously throughout the entire saturated flow domain, by using a series expansion of the velocity potential (Powers et al, 1967; Powell and Kirkham, 1976). In the past, series solutions have been severely restricted by the necessity for a horizontal lower boundary. However, the least squares methodology has been applied to the problem (Read and Volker, 1990; Volker and Read, 1990), enabling series expansions for arbitrary bottom geometry to be obtained.

Unfortunately, accurate numerical solutions for mass transport can be extremely difficult to obtain, due to numerical dispersion. This problem is compounded in the hillslope context, where large aspect (i.e. length to depth) ratios of 50:1 or 100:1 are common, and any numerical solution, whether accurate or not, is usually obtained at a high computational cost. Analytical solutions for contaminant transport are available for infinite flow domains, (Hunt, 1983) when the seepage velocity is constant. Although solutions of this type are not directly applicable to the hillside problem, they avoid the common pitfalls encountered by numerical schemes.

The simplified soil horizon for a general hillside seepage problem is depicted in Figure 1. For large aspect ratios (greater than 20:1), Volker and Read (1990) have shown that the seepage velocity is uniform for most of the saturated flow domain. Based on this feature of hillside seepage, and using a series solution for the seepage velocity, this paper develops a method for transforming the contaminant transport problem to a square flow domain, with coordinate axes parallel and perpendicular to the direction of the seepage velocity. An analytical solution on this finite domain is then obtained for the solute concentrations, using a series expansion.

Section 2 provides a formal mathematical description of the hillside seepage problem, while Section 3 outlines the least squares method used to obtain the series expansion for the velocity potential. Section 4 details the transformation process and the series expansion for the solute concentrations, while some representative analyses are presented in Section 5. Finally, the method and results are discussed in Section 6.

2 FORMAL PROBLEM DESCRIPTION

Solutions for the flow equation and the transport equation are necessary to obtain solute concentrations throughout the flow field. The flow and transport equations must be solved subject to the imposed boundary conditions, on the flow domain (Figure 1) and the transformed flow domain (Figure 2), respectively.

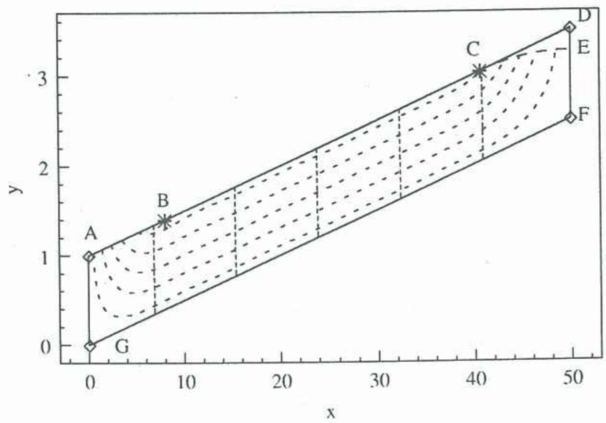

Figure 1
The hillside seepage flow domain. Dotted sloping lines are streamlines, indicating 10%, 30%, 50%, 70%, and 90% of the flow. Vertical dotted lines are equipotential lines, indicating 10%, 30%, 50%, 70%, and 90% of the potential range.

In Figure 1, AGFD forms the impermeable boundary, consisting of the vertical boundaries AG, FD and the sloping lower boundary GF. Fluid moves from the unsaturated zone above the water table CE, through the saturated aquifer and seeps through the lower section of the soil surface, AB. The seepage face AC consists of two sections, separated by the stagnation point B. Fluid flows across the seepage face AB, while the fluid velocity is essentially parallel to section BC: that is, there is no inflow or seepage across the soil surface BC. Technically speaking, the seepage face consists of AB,

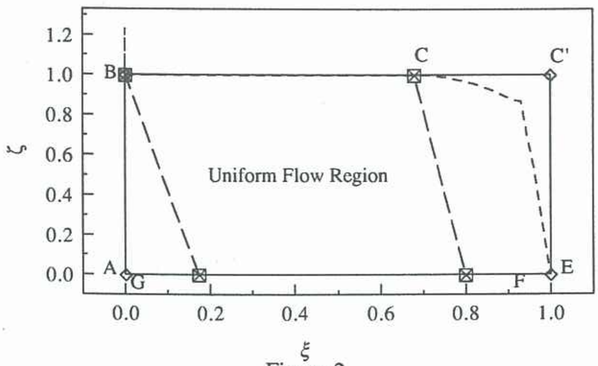

Figure 2
The transformed flow domain. The sloping dotted lines delineate the boundaries of the uniform flow region, while the dotted line CE indicates the transformed water table

not AC. By convention, however, the upper limit of the seepage face is taken as the intersection of the water table and the soil surface.

The transformation to the curvilinear coordinate system (ξ,ζ) maps the points A, B, C, E, F, G in Figure 1 to the corresponding points in Figure 2. The ξ coordinate is measured along a streamline (parallel to the direction of fluid flow), starting at the outflow section of the seepage face (AB in Figure 1). The ζ coordinate is measured along the equipotential lines (perpendicular to the direction of fluid flow), starting at the impermeable boundary (AGFE in Figure 1).

2.1 The Flow Equation

Seepage through saturated, permeable soil is governed by Darcy's Law. Assuming the soil is homogeneous and isotropic, the velocities U,V in the x,y directions are given by

$$U(x,y) = -K\,\partial\phi/\partial x, \qquad V(x,y) = -K\,\partial\phi/\partial y \tag{1}$$

where $\phi(x,y)$ is the hydraulic potential and K is the constant hydraulic conductivity. For an incompressible fluid, application of the continuity condition leads to Laplace's equation, in the fully saturated zone ABCEFG (Figure 1):

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0 \tag{2}$$

The boundary conditions along the saturated flow perimeter consist of a combination of velocity and potential forms. Along impermeable boundaries AG, GF, DF, the normal velocity is zero:

$$\partial\phi/\partial m = 0 \tag{3}$$

where m is normal to the boundary. When the impermeable boundary is vertical (AG, DF), this condition reduces to:

$$\partial\phi/\partial x = 0 \tag{4}$$

The upper saturated flow boundary $y_t(x)$ (ACE) is permeable, and consists of a seepage face $f_t(x)$ (AC) and water table $\eta(x)$ (CE) For steady recharge along the water table, the surface potential along the seepage face ($y_t(x)$, $0 \le x \le x_L$) and the water table ($y_t(x)$, $x_L < x \le s$) is equal to the elevation above an arbitrary datum:

$$\phi_t(x) = y_t(x) = \begin{cases} f_t(x) & , 0 \le x < x_L \\ \eta(x) & , x_L \le x \le s \end{cases} \tag{5}$$

where x_L is the x coordinate of the intersection of the seepage face, and s the horizontal base length. For recharge $r = K.R$ along the water table, conservation of mass stipulates that

$$R = \partial\phi/\partial y - \eta'(x).\partial\phi/\partial x \tag{6}$$

These boundary conditions applied to the equation of continuity, fully define the steady seepage through the hillslope.

2.2 The Transport Equation

The equation for solute transport in the saturated region ABC'E (Figure 2) of the transformed flow domain can be expressed as (Hunt, 1983):

$$\frac{\partial}{\partial\xi}[D_\xi\frac{\partial C}{\partial\xi}] + \frac{\partial}{\partial\zeta}[D_\zeta\frac{\partial C}{\partial\zeta}] - u\frac{\partial C}{\partial\xi} = \frac{\partial C}{\partial t} \tag{7}$$

where $C(\xi,\zeta,t)$ is the concentration of the solute (mass per unit volume), D_ξ, D_ζ are the components of the dispersion tensor D, and u is the magnitude of the pore velocity. The dispersion tensor D is given by

$$D_\xi = \alpha_\xi u\ ,\ D_\zeta = \alpha_\zeta u \tag{8}$$

where α_ξ is the longitudinal dispersivity in the direction of fluid velocity, α_ζ is the lateral dispersivity normal to the direction of the fluid velocity. Note that the dispersivities α_ξ and α_ζ are constant with respect to time and position.

The boundary conditions for solute transport need to be described along the impermeable boundary (AE), seepage face (AB and BC') and water table (C'E). At the lower impermeable boundary AE there is no flux across the boundary; as the normal velocity is zero, the boundary condition for $C(\xi,\zeta,t)$ along AE is

$$D_\zeta\frac{\partial C}{\partial\zeta} = 0 \tag{9}$$

Along BC', the normal component of the velocity is zero, and the soil surface acts as a boundary between the saturated region and the atmosphere. Modelling the atmosphere as a gas continuum (Bear, 1979), with no dispersion across the boundary, the boundary condition reduces to Eq. 9.

Along the water table CE, the boundary acts a division between unsaturated and saturated flow regimes. Assuming the unsaturated zone can be modelled as a gas continuum, the net flux across the water table will be due to advection only. Similarly, the seepage face acts as a boundary between the saturated zone and the atmosphere, and once again the net flux will be due to advection only. Hence, the boundary conditions along AB and CE reduce to (Bear, 1979):

$$D_\xi\frac{\partial C}{\partial\xi} = 0 \tag{10}$$

Initially, there is no solute in the saturated flow region, apart from a rectangular region of constant concentration C_0 downstream from the water table. At $t = 0$, the concentration $C(\xi,\zeta,0) = C^0(\xi,\zeta)$ is given by

$$C^0(\xi,\zeta) = C_0,\ \xi_0 \le \xi \le \xi_1,\ \zeta_0 \le \zeta \le \zeta_1 \\ = 0, \text{ otherwise.} \tag{11}$$

These initial and boundary conditions applied to the solute transport equation fully define the transport process.

3 SERIES SOLUTIONS FOR SEEPAGE

An analytical solution for the hydraulic potential and hence the seepage velocity can be obtained by applying the method of separation of variables. The appropriate truncated form of the series expansion, for the given boundary conditions is:

$$\phi(x,y) = c_0 + \sum_{n=1}^{N} \left(a_n e^{\frac{n\pi y}{s}} + b_n e^{\frac{-n\pi y}{s}}\right)\cos\frac{n\pi x}{s} \qquad (12)$$

The bottom boundary condition (Eq. 3) determines the relationship between a_n and b_n. This boundary condition can be expressed concisely as

$$\bar{\phi}_b(x) = 0.c_0 + \sum_{n=1}^{N} \bar{u}_n^b(x)a_n + \sum_{n=1}^{N} \bar{v}_n^b(x)b_n = 0 \qquad (13)$$

where

$$\bar{u}_n^b(x) = \tfrac{\pi}{s}n\left(\cos\frac{n\pi x}{s} + \dot{f}_b(x)\sin\frac{n\pi x}{s}\right)e^{\frac{n\pi f_b(x)}{s}}$$

and (14)

$$\bar{v}_n^b(x) = -\tfrac{\pi}{s}n\left(\cos\frac{n\pi x}{s} - \dot{f}_b(x)\sin\frac{n\pi x}{s}\right)e^{\frac{-n\pi f_b(x)}{s}}$$

The two sets of basis functions $\bar{u}_n^b(x)$ and $\bar{v}_n^b(x)$ form a linearly dependent spanning set, and the least squares method can be used to eliminate this dependence. After some simplification, the relationship between the basis functions can be represented by an $N{\times}N$ system of equations in (constant) h_{ik};

$$\int_0^s \bar{u}_i^b(x)\bar{v}_j^b(x)dx = \sum_{k=1}^{N} h_{ik}\int_0^s \bar{v}_k^b(x)\bar{v}_j^b(x)\,dx \qquad (15)$$

This set of $N{\times}N$ simultaneous equations can be solved for h_{ij}.

The series expansion for the hydraulic potential can now be expressed concisely as

$$\phi(x,y) = \sum_{n=0}^{N} c_n w_n(x,y) \qquad (16)$$

where $w_0(x,y) = 1$,

and (17)

$$w_n(x,y) = \left(e^{\frac{n\pi y}{s}} - \sum_{k=1}^{N} h_{nk}e^{\frac{-n\pi y}{s}}\right)\cos\frac{n\pi x}{s}, \text{ for } n = 1,\ldots,N.$$

Once again, the least squares technique can be used to evaluate the coefficients c_n, by applying the top boundary condition for $y_t(x)$ (Eq 5). Minimisation of the squared error in the top boundary leads, after some simplification, to $(N+1)$ equations of the following form:

$$\int_0^s \phi_t(x)w_j^t(x)\,dx = \sum_{n=0}^{N} c_n\int_0^s w_n^t(x)w_j^t(x)dx \qquad (18)$$

The sub- and super-script t refers to the appropriate function on the top boundary $y_t(x)$. This set of $(N+1)$ simultaneous equations can be solved for the c_n, after a slight modification to the surface potential.

The surface potential $\phi_t(x)$ has a small discontinuity at the downstream impermeable boundary (Klute et al, 1965), which can be removed by using a cubic spline to take the surface potential smoothly to zero slope at $x = 0$, without appreciably affecting the surface potential.

This solution technique for the series coefficients c_n depends on the initially unknown water table location $\eta(x)$. The boundary condition (Eq 6) that determines $\eta(x)$ is both nonlinear and implicit, excluding any exact representation for the water table. Numerical schemes such as the boundary integral method overcome this problem by assuming an initial position, and then iteratively improving it using the appropriate boundary condition. Using a similar approach for the present method, the water table is approximated by a cubic spline, and the approximation iteratively improved using Eq 6.

4 SERIES SOLUTIONS FOR SOLUTE TRANSPORT

The contaminant transport problem in the (x,y) coordinate system is transformed to the (ξ,ζ) coordinate system, by using the series expansion for the stream function and the potential function. The stream function is obtained from the potential function by invoking the Cauchy-Riemann equations:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}\ , \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \qquad (19)$$

The (x,y) coordinates are then transformed to (ξ,ζ) coordinates, by integrating the distance along the stream- and equipotential lines, whose equations are given by the following implicit relationships:

$$\psi(x,y) = \text{const.}\ , \quad \phi(x,y) = \text{const} \qquad (20)$$

After the stream length and equipotential length have been calculated, they are normalised, using the maximum stream length and maximum equipotential length.

Once the solution is obtained in the (ξ,ζ) domain, the inverse transformation is achieved by reversing this process. The x coordinate can be determined approximately along the bottom boundary ($\zeta = 0,$) and the distance integrated along the equipotential line, to height ζ. This estimate can be improved iteratively, by integrating along the stream- and equipotential lines using the estimated (x,y) coordinates. The estimated and true values of (ξ,ζ) are then used to obtain better estimates of (x,y).

The solution to the transport equation in the (ξ,ζ) domain is obtained by first transforming the transport equation (Eq. 7)to the heat equation. Application of separation of variables to the resulting differential equation produces two ordinary differential equations. Both equations are in self adjoint form, and the problem is reduced to solving two Sturm-Louiville eigenvalue and eigenfunction problems. The first boundary condition (Eq. 9) leads to an infinite number of negative eigenvalues, whereas the second boundary condition (Eq. 10) produces one positive and an infinite number of negative eigenvalues. The solution for the concentration $C(\xi,\zeta,t)$ is given by

$$C(\xi,\zeta,t) = \sum_{m=0}^{\infty} A_{\lambda m}\cos m\pi\zeta\, e^{-\mu t} +$$

$$e^{\nu}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} A_{nm}\left(\sin n\pi\xi - \frac{2n\pi D_\xi}{u}\cos n\pi\xi\right)\cos m\pi\zeta\, e^{-\gamma t} \qquad (21)$$

where $\mu = m^2\pi^2 D_\zeta$, $\nu = u(2\xi - ut)/4D_\xi$ and $\gamma = n^2\pi^2 D_\xi + m^2\pi^2 D_\zeta$.

The coefficients A_{nm} are evaluated from the initial conditions (Eq. 11), by using the orthogonality relationship, after some modification for the positive eigenvalue case.

5 RESULTS AND DISCUSSION

Solutions for solute transport have been obtained for a hillslope of the form shown in Figure 1, with two sets of dispersion coefficients D_ξ and D_ζ. The flow domain consists of sloping, parallel upper and lower boundaries (slope 0.05) with an aspect ratio of 50:1, as shown in Figure 4. The hillslope is subject to an applied recharge of $5.0{\times}10^{-3}K$, producing a steady water table and seepage velocity $5.0{\times}10^{-2}K$ in the uniform flow region. A rectangular block of solute with initial concentration C_0 is located below the water table in the (ξ,ζ) domain, roughly beneath the intersection of

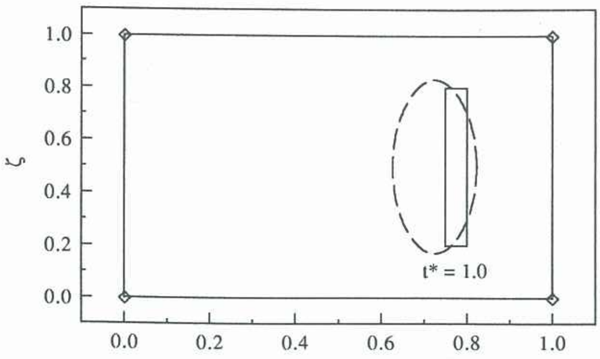

Figure 3

Solute transport in the (ξ,ζ) domain with (α_x,α_y)=(5,0.01).

the water table and the soil surface, as shown in Figure 3.

The transformation process involves normalising the length and depth scales. Additionally, the pore velocity is used for solute transport, whereas the seepage velocity is given by the solution to the flow equation. The relationship between the two flow domains can be described, by introducing dispersion parameters D^*_ξ, D^*_ζ and time parameter t^* in the (ξ,ζ) domain. The relationship between these parameters and the corresponding parameters in the (x,y) domain is given by

$$D^*_\xi = \theta D_x/lK,\ D^*_\zeta = \theta l D_y/d^2K,\ t^* = Kt/\theta l \tag{22}$$

where θ is the soil porosity, l is the maximum stream length, d is the maximum equipotential length and D_x, D_y are the dispersion coefficients in the (x,y) domain.

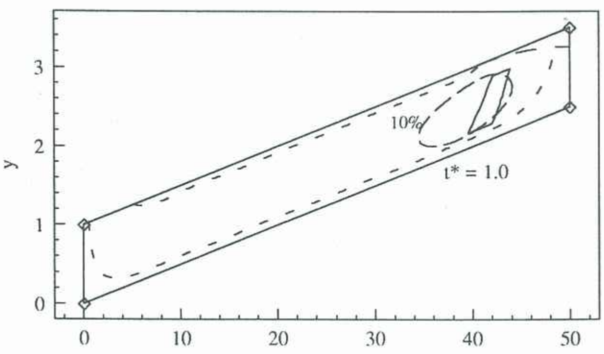

Figure 4

Solute transport in the (x,y) domain with (α_x,α_y)=(5,0.01).

The dispersivities in the (x,y) domain are given in terms of the dispersivities α_x,α_y and the pore velocity u (Eq. 8). In terms of the dispersivities and the seepage velocity u_sK, the dispersion coefficients D^*_ξ, D^*_ζ reduce to

$$D^*_\xi = \alpha_x u_s/l\ ,\ D^*_\zeta = \alpha_y l u_s/d^2 \tag{23}$$

The analyses presented show the solute concentrations in the form of contours of dimensionless concentration (relative to C_0) at varying times with two sets of dispersion coefficients. Figures 3 and 4 show the solutions in both flow domains for (α_x,α_y) = (5.0, 0.01). For these parameters, the longitudinal dispersion process dominates, so much so that the concentration of the solute drops below 10%, shortly after t^*=1. Figures 5 and 6 show the solutions in both flow domains for (α_x,α_y) = (1.0, 0.001). For this example, the dispersion process is not dominant, with advection carrying the solute through most of the soil before the concentration drops beneath 10%, shortly after t^*=12. In both cases, however, the solute concentrations drop sharply from the initial 100% concentration to below 50%.

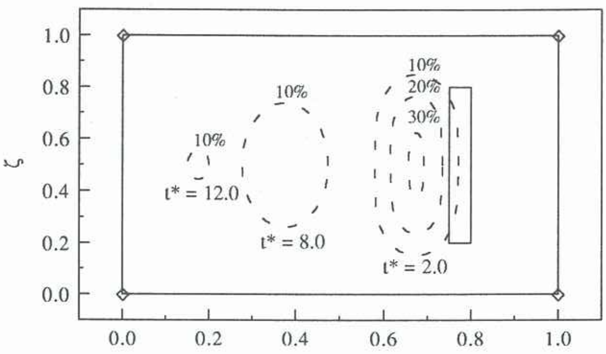

Figure 5

Solute transport in the (ξ,ζ) domain with (α_x,α_y)=(1,0.001).

6 CONCLUSIONS

Series solutions have been developed for solute transport in hillside seepage, based on series solutions for the velocity field. The solutions, although not exact for the entire flow domain, provide an efficient means of estimating the extent of solute transport. Accurate numerical solutions of the flow equation are difficult to obtain, without large computational expense, for the long aspect ratios characteristic of hillside seepage. These series solutions overcome that deficiency and also show that the flow is close to uniform over the majority of the flow domain. This means that a transformation of the domain into equipotential and streamline coordinates produces a result ideally suited to analytical solutions of the transport equation.

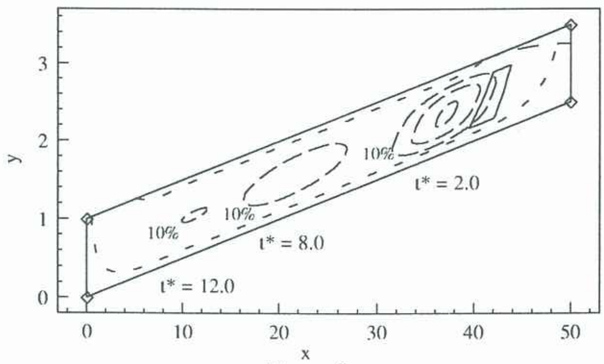

Figure 6

Solute transport in the (x,y) domain with (α_x,α_y)=(1,0.001).

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