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ON THE FLOW PAST A SPHERE ON THE AXIS OF A ROTATING FLUID

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ABSTRACT

This paper examines the flow when a sphere moves at a constant velocity up the axis of a uniformly rotating fluid. This is a classical problem in fluid dynamics, originally examined in a series of experimental papers (Taylor, 1922). Subsequent theoretical work has concentrated on the case where the flow is assumed to be slow enough that inertial effects can be neglected and the resulting linear equations are solved either as an unsteady initial-value problem or with viscosity included. In this study the equations for the slightly viscous problem are to be solved numerically and the results will be compared with the experimental results for this case by Maxworthy (1970). This will enable a more detailed examination of the flow than was possible in the experimental study, due to visualisation difficulties.

INTRODUCTION

In the early 1920's G.I. Taylor performed a series of experiments on flow in a rapidly rotating reference frame, where a body of fluid was placed in rigid rotation and then perturbed by moving an obstacle relatively slowly (compared with the rotational speeds) through the fluid. In one of these experiments (Taylor, 1922), a ping-pong ball was placed in a container of fluid which was then rotated at a constant angular velocity until the entire system was in rigid rotation. The ball was then pulled downwards at a uniform speed through the container by an attached thread. It was found that a column of fluid, with the same diameter as the sphere, was pushed along in front of the sphere.

Taylor's paper also presented a family of solutions for the steady inviscid problem, satisfying the solid surface condition on the sphere and tending a uniform flow at infinity. Taylor also recognised that some of these solutions would be unlikely to occur in a steady solution resulting from an initial value problem in which the sphere was started from rest. A further difficulty with the solutions was that they represented a large disturbance to the motion, even if the sphere was moving very slowly.

Subsequent work attacked from the problem by two different approaches, but mostly under the assumption that the motion induced by the sphere was relatively small, that is that the Rossby number in the problem is small. Under these conditions the governing equations are linear. The two approaches were to consider either the unsteady inviscid problem, or the steady viscous problem.

The unsteady problem of a sphere brought into uniform motion impulsively from rest was first examined by Grace (1926), who examined the solution for small times and attempted to estimate the steady drag by extrapolation. Stewartson (1952) reconsidered the same problem and derived an analytical solution, written in terms of a Laplace transform. For large times he outlined the asymptotic value for the drag and the form of the steady solution. In this solution the axial velocity is constant within the axial cylinder enclosing the sphere, with a decreased swirl ahead of the sphere and increased swirl behind. A more detailed examination of the unsteady solution was examined for a related problem by Bretherton (1967).

Nonlinear effects for the inviscid problem have been considered in papers by Long (1953) and Stewartson (1958), based on the so-called "Long's model" where a quantity related to the vorticity is conserved along streamlines (under the assumption of an undisturbed flow upstream). In particular, Stewartson examined the conditions under which a body of fluid was pushed ahead of the sphere and concluded that this does not occur when the Rossby number is large enough.

The viscous problem was apparently considered originally by Morrison & Morgan (1956), although the author has been unable to obtain a copy of this report. Further work was presented for the case of a sphere in a finite length container by Moore & Saffman (1968) and later, in more detail and including the case of an unbounded fluid, by Moore & Saffman (1969). In these two papers the solution was derived under the assumption that the Ekman number, which represents the importance of viscous effects, was small by using an asymptotic analysis. In particular, they noted that the scale length of the upstream 'blocked' region was inversely proportional to the Ekman number. Hocking, Moore & Walton (1979) reconsidered the same problem for 'long' containers and noted the modifications in the drag on the sphere due to the end walls, even when they are very far from the sphere.

The results from these theoretical papers were supported by two experimental papers by Maxworthy (1968, 1970). The first paper, which considers short containers, is of less interest here than the second paper which demonstrated the presence of a variety of phenomena in this problem. It is quite clear that both nonlinear and viscous effects were important in these flows yet to date no detailed numerical calculations have been performed for this configuration in order to examine details of the flow. This is particularly important since the experiments were unable to reveal fine details of the flow structure due to the presence of the strong swirling motion which mixes any dye which is released. The only numerical work which has been performed for this problem, by Dennis, Ingham & Singh (1982), is for a very viscous flow.

The following sections present details of the governing equations for this flow and the numerical technique used to obtain solutions of them for the parameter regime considered in Maxworthy (1970). At this stage only preliminary solutions are available but they reveal the presence of three important phenomena in this problem: upstream influence, viscous separation and steady downstream waves.

GOVERNING EQUATIONS

This paper concerns the motion of a homogeneous viscous fluid, with constant kinematic viscosity ν^*, in a rotating fluid moving with constant angular velocity Ω^* far from the spherical obstacle of radius l^*. The sphere rotates with the same angular velocity and moves parallel to the axis of rotation at a uniform velocity U^*. The motion of the fluid is observed from a reference frame rotating with the fluid and moving with the sphere.

Two nondimensional parameters, the Rossby number $Ro = U^*/\Omega^* l^*$ and Ekman number $E = \nu^*/\Omega^* l^{*2}$, can be defined (Greenspan, 1968) and these determine the flow in a nondimensional coordinate system based on the length scale l^*, time scale $1/\Omega^*$ and velocity scale U^*. Cylindrical polar coordinates are used here, aligned with the axis of rotation and with the origin at the centre of the sphere, and the flow is assumed to be axisymmetric.

A stream function ψ is introduced so that the velocity components (u, w), in the (r, z) directions, are given by

$$u = \frac{1}{r}\frac{\partial\psi}{\partial z} \quad \text{and} \quad w = -\frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{1}$$

The equations of motion for the unsteady flow are then

$$\frac{\partial\zeta}{\partial t} + RoJ(\psi, \frac{\zeta}{r}) - 2\overline{\Omega}\frac{\partial v}{\partial z} = E[\nabla^2\zeta - \frac{\zeta}{r^2}] \tag{2}$$

$$\frac{\partial v}{\partial t} + RoJ(\psi, \frac{v}{r}) + 2\frac{\overline{\Omega}}{r}\frac{\partial\psi}{\partial z} = E[\nabla^2 v - \frac{v}{r^2}] \tag{3}$$

where

$$J(\psi, *) = \frac{\partial\psi}{\partial z}\frac{\partial *}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial *}{\partial z}, \qquad \overline{\Omega} = 1 + Ro\frac{v}{r} \tag{4}$$

and ∇^2 is the usual Laplacian in cylindrical coordinates. Here ζ is the azimuthal vorticity component

$$\zeta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} = \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\psi}{\partial r}) + \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\psi}{\partial z}) \tag{5}$$

and v is the azimuthal velocity component, also known as the swirl velocity. The boundary conditions applied on these equations are that (u, w) tends to $(0, 1)$ and $\partial v/\partial z$ tends to zero for $r^2 + z^2 \gg 1$, and that $u = w = 0$ on the sphere at $r^2+z^2 = 1$. The initial trials have been performed with $v = 0$ on the cylinder but, for direct comparison with the experiments, the rotation rate of the sphere should be such that the total azimuthal torque is zero.

The parameter regime of interest in this study is that with $Ro \ll 1$ and $E \ll 1$, corresponding to slowly-moving sphere in a rapidly-rotating viscous fluid. This is similar to that examined in the experiments by Maxworthy (1970) and as Ro tends to zero it corresponds to the viscous theory examined by Moore & Saffman (1969).

NUMERICAL METHOD

The numerical technique used to solve the coupled nonlinear equations (2), (3) and (5) is based on a finite-difference approximation of the equations in a conformally-transformed spatial domain. The region $r^2 + z^2 \geq 1$ with $r \geq 0$ is transformed into a semi-infinite region (x', y') with $y' \geq 0$ using the Joukowski transformation

$$x' + iy' = f(z + ir) = z + ir + \frac{1}{z + ir} \tag{6}$$

in terms of complex-valued variables. The advantage of this form of transformation is that the partial differential equations are relatively unchanged in terms of the (x', y') coordinates. Far from the sphere f approaches an identity transformation. One-dimensional grid stretching was used in each of x' and y' to improve resolution near the sphere at $-2 \leq x' \leq 2$ and $y' = 0$. The flow was solved in a rectangular region for (x', y') with a variety of boundary and radiation conditions applied at the outer edges of the domain.

An alternating-directions-implicit technique was used to solve (2) and (3), with either centred or third-order upwind differencing used for the advective terms. These equations were solved in conjunction with the elliptic equation (5) for the stream function, with an iteration for the nonlinear terms at each time step. Equation (5) is nonseparable in the transformed coordinates and was solved using the general multigrid routine MGD9V (de Zeeuw, 1990). The set of equations were integrated from initial conditions corresponding to an impulsive start to the motion at $t = 0$, with $\zeta = v = 0$ in most of the flow, and the integrations continued until $\partial\psi/\partial t$ was less than 0.1%.

RESULTS

In figure 1 some preliminary results are presented for the stream function ψ, vorticity ζ and swirl velocity v when $Ro = 0$ and $E = 10^{-3}$. The streamlines in the plot for ψ are relative to the motion of the sphere and they clearly demonstrate that the flow ahead and behind the sphere is slow, compared to that with $r > 1$. In the context of the experiment this is means that the fluid with $r < 1$ is pulled along with the sphere, as noted by both Taylor and Maxworthy. The plot of the vorticity ζ shows a region of high vorticity on the surface of the sphere (the Ekman layer, of thickness $O(E^{\frac{1}{2}})$ according to linear theory) and a less intense region of vorticity extending along the line $r = 1$. This later region corresponds to the Stewartson layers (which are thicker than the Ekamn layer), as noted by Moore & Saffman (1969). That paper also noted the presence of a singularity in the Stewartson-layer flow at $(r, z) = (1, 0)$, and this is apparent in the numerical results as a region of high velocity.

The swirl velocity v in figure 1 is clearly concentrated in the region $r < 1$, bounded by the sphere diameter, with virtually no swirling motion outside of this cylinder. This is

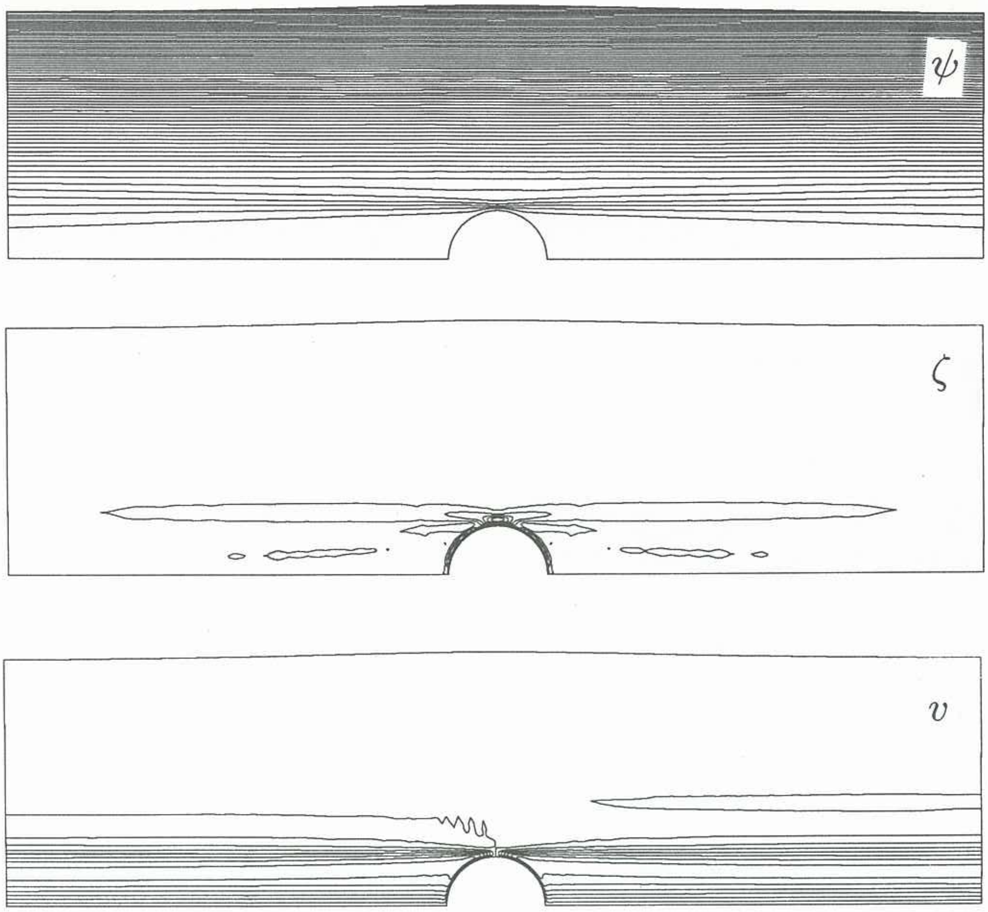

Figure 1. - Numerical results for the flow past a sphere when $Ro = 0$ and $E = 10^{-3}$. The solution is plotted over the range $r \leq 10$ and $-10 \leq z \leq 10$. Contour intervals are $\Delta\psi = 0.2$, $\Delta\zeta = 1$ and $\Delta v = 1$.

in agreement with both the experimental observations and the theoretical results by Stewartson (1952) which suggest that

$$v \sim \pm\frac{2r}{\pi\sqrt{1-r^2}} \quad \text{for } r < 1 \qquad (7)$$

with $v < 0$ ahead of the cylinder and $v > 0$ behind the cylinder. In particular, the singularity in this expression, which is smoothed out across the Stewartson layers in a viscous fluid, is apparent in the numerical results by a maximum in v near $r = 1$.

For $Ro > 0$ the flow is no longer symmetric about $z = 0$ and preliminary results show the presence of small-amplitude stationary lee waves just behind the cylinder and, for larger Rossby numbers, the separation of the Ekman layer in the same region. The loss of symmetry about $z = 0$ means, among other things, that the net torque on the sphere will be non zero and therefore the numerical results with $v = 0$ on the cylinder will not correspond exactly to those in the experiments. However, since the same features were present in the experiments it is not expected that a change in the boundary condition will affect the qualitative features of the flow. Similar features were also present in another study, using an earlier version of the code, for flow past a circular cylindrical obstacle (or, equivalently, flow over a cylindrical obstacle in a Boussinesq stratified fluid).

Further trials and comparisons with the experimental and theoretical results, which are available in special limits, are continuing and a more comprehensive set of results will be available at the time of the conference. Aspects to be pursued in detail include the drag on the cylinder and the occurrence of hydraulic jump type phenomena, possibly related to vortex breakdown in the region behind the sphere. A more detailed examination of the effect of different types of radiation conditions will also be conducted.

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REFERENCES

BRETHERTON, F.P. (1967) The time-dependent motion due to a cylinder moving in an unbounded rotating or stratified fluid. *J. Fluid Mech.*, **28**, 545-570.

DENNIS, S.C.R. & INGHAM, D.B. (1982) The slow translation of a sphere in a rotating viscous fluid. *J. Fluid Mech.*, **117**, 251-267.

GRACE, S.F. (1926) One the motion of a sphere in a rotating liquid. *Proc. Roy. Soc. Lond. A*, **113**, 46.

GREENSPAN, H.P. (1968) The theory of rotating fluids. Cambridge University Press.

HOCKING, L.M., MOORE, D.W. & WALTON, I.C. (1979) The drag on a sphere moving axially in a long rotating container. *J. Fluid Mech.*, **90**, 781-793.

LONG, R.R. (1953) Steady motion about a symmetrical obstacle moving along the axis of a rotating fluid. *J. Met. Soc. Japan*, **10**, 197.

MAXWORTHY, T. (1968) The observed motion of a sphere through a short, rotating cylinder of fluid. *J. Fluid Mech.*, **31**, 643-655.

MAXWORTHY, T. (1970) The flow created by a sphere moving along the axis of a rotating, slightly-viscous fluid. *J. Fluid Mech.*, **40**, 453-479.

MOORE, D.W. & SAFFMAN, P.G. (1968) The rise of a body through a rotating fluid in a container of finite length. *J. Fluid Mech.*, **31**, 635-641.

MOORE, D.W. & SAFFMAN, P.G. (1969) The structure of free vertical shear layers in a rotating fluid and the motion produced by a slow rising body. *Phil. Trans. Roy. Soc Lond. A*, **264**, 597-634.

MORRISON, J.W. & MORGAN, G.W. (1956) The slow motion of a disk along the axis of a viscous, rotating fluid, Report 56207/8, Div. Appl. Math. Brown University.

STEWARTSON, K. (1952) On the slow motion of a sphere along the axis of a rotating fluid. *Proc. Camb. Phil. Soc.*, **48**, 168-177.

STEWARTSON, K. (1958) The motion of a sphere along the axis of a rotating fluid. *Quart. J. Mech. Appl. Math.*, **11**, 39-51.

TAYLOR, G.I. (1922) The motion of a sphere in a rotating liquid. *Proc. Roy. Soc. Lond. A*, **102**, 180-189.

de ZEEUW, P.M. (1990) Matrix-dependent prolongations and restrictions in a blackbox multigrid solver. *Jour. Comp. Appl. Maths.*, **33**, 1-27.