11th Australasian Fluid Mechanics Conference
University of Tasmania, Hobart, Australia
14-18 December 1992

POTENTIAL FLOW ANALYSIS OF FLOW OVER A CURVED BROADCRESTED WEIR

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SUMMARY

Flow over a curved broadcrested weir is analysed by a potential flow method of the so-called "inverse type". This method allows the numerical solution of the free boundary problem, by using an iterative algorithm to adjust the position of the free surface. The method advocated here is proven to be fast and efficient and the results compare favourably with data from the careful experiments of Blau (1962).

INTRODUCTION

1.1 Although in many cases it is feasible to determine the flow characteristics of control weirs by recourse to model testing, it is of interest to compare the empirical results with the theoretical calibration of such weirs, as deduced from potential flow analysis. This analysis supplies not only the coefficient of discharge, but also a very complete information on the velocity and pressure fields about the weir.

A useful application of the potential flow analysis is the assessment of the effect of small changes in boundary geometry on the pressure and velocity fields about a curved weir, such as those commonly used in the control of open channel flow and on dam spillways.

Many of the previous solutions for flows about curved weirs are confined to the characteristics on the crest region (Jaeger, 1955, Montes, 1970). More general methods have been developed, all of them based on the inviscid flow approximation that leads to an elliptic boundary value problem. Thus, for example, Strelkoff (1970) proposed a method that reduced the problem to the solution of an integral equation which was then numerically solved. Dao-Yang and Man-Ling (1979) used the Finite Element Method to solve the problem of flow over a standard spillway while Cheng, Liggett and Liu (1981) used the Boundary Element method to analyse a similar case.

The older approach of solving Laplace's equation by relaxation in the real x, y plane was fraught with practical problems due to the curvilinear boundaries of the flow. Thom and Apelt (1961) showed that the finite-difference solution by relaxation in the interior of a flow domain such as that shown in figure 1 can be considerably simplified if the relaxation procedure is implemented in the complex potential plane $\phi-\psi$ plane where the boundaries are rectangular, avoiding therefore the problem of irregular computational stars.

Cassidy (1967) used this approach in the solution of the flow over a standard spillway. It was also used by Markland (1965) in relation with the problem of flow on a free overfall.

1.2 All the previously discussed methods assume that at any instant a free surface profile has been defined, along which a boundary condition (in this case constant pressure) determines the dependent variable values.

Subject to this assumption, these methods satisfy Laplace's equation for ϕ or ψ in the interior of the flow domain together with the usual zero normal velocity condition on the impervious boundaries.

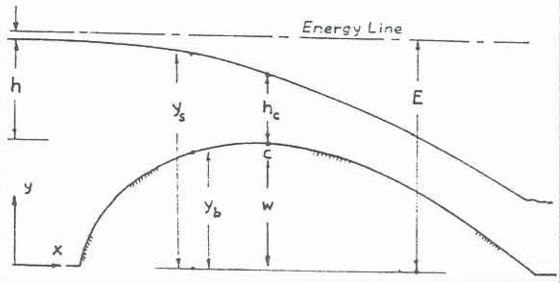

Figure 1 Geometry of flow over a curved weir

The solution process is carried out iteratively: an initial free surface profile is assumed, then Laplace's equation is solved in the interior of the flow domain thus defined, and the condition of constant pressure along the free surface is tested. If this has not been achieved, the free surface is moved to a new location, needing a fresh solution of the boundary value problem. This two stage cycle of free surface relocation and Laplace equation solution is continued until the pressure condition is satisfied within a prescribed tolerance along the whole free surface.

Because of the non-linearity of the free surface condition,the algorithms used in the search for a definitive free surface are sensitive to small differences between the true and the local value of the calculated energy.This rules out any attempt to start the iterative process with arbitrary initial free surface profiles, but one can, in many cases prescribe an initial trial surface based on a simplified one dimensional energy approach.

2. THE $x-\psi$ METHOD:

2.1 In Thom and Apelt's (1961) inverse method, the role of the dependent (ϕ, ψ) and independent set of variables (x, y) were interchanged, resulting in a Laplace equation for y that reads:

$$\frac{\partial^2 y}{\partial \phi^2} + \frac{\partial^2 y}{\partial \psi^2} = 0 \tag{1}$$

The $x-\psi$ method may be viewed as belonging to the same class of variable interchange methods, but only one variable is exchanged. In this case y and ψ change their respective roles, so that ϕ, y becomes functions of x and ψ.

This method was proposed by Boadway (1976), and has been used with some modification by Crank and Ozik (1980) for the solution of flow in porous media and by Montes (1992) for the solution of the free overfall case. Boadway found that in terms of the new independent variables, x and ψ, Laplace's equation for the depth y may be written as:

$$\frac{\partial^2 y}{\partial x^2}\left[\frac{\partial y}{\partial \phi}\right]^2 + \frac{\partial^2 y}{\partial \psi^2}\left[1 + \left[\frac{\partial y}{\partial x}\right]^2\right] - 2\frac{\partial^2 y}{\partial y \partial \psi}\frac{\partial y}{\partial x}\frac{\partial y}{\partial \psi} = 0 \tag{2}$$

As it is easily realised, the variable interchange maps the real plane into a rectangular strip in the x-y plane, as indicated in figure 2. In this plane, the velocities u and v are determined by:

$$u = \frac{1}{\dfrac{\partial \psi}{\partial y}} \quad \text{and } v = u\frac{\partial y}{\partial x} \tag{3}$$

A restriction of the method is that there cannot be a persistent 90^0 change in boundary alignment, as there the derivative dy/dx in (2) becomes indeterminate, and y = y(x) multivalued.

2.2 The solution of y(x) that satisfies (2) is subject to the following boundary conditions: at the solid boundary FDC, the boundary shape y = y(x) is prescribed. By the previous arguments, no reentrant corners are allowed. at the free surface AC'B, taking p = 0, y is given by: y_s = E –

$$\frac{V^2}{2g} = E - \left[1 + \left[\frac{\partial y}{\partial x}\right]^2\right] / 2g\left[\frac{\partial y}{\partial \psi}\right]^2 \tag{4}$$

Along the vertical boundaries AF, BD, it is assumed that the horizontal velocity u is uniformly distributed. This assumption is acceptable if sections AF and BC are in regions of slow change of boundary shape. This condition leads to a linear variation of y with the stream function

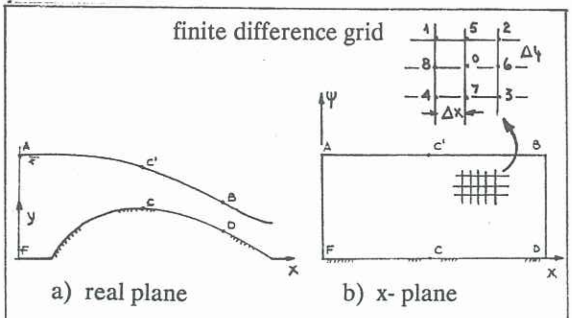

Figure 2 Boundary transformation

$$y(\psi) = y_b + (y_s - y_b)\frac{\psi}{q} \tag{5}$$

where y_s and y_b are the surface and bottom ordinates.

3. NUMERICAL METHOD OF SOLUTION

3.1 As the flow domain maps into a rectangular strip in the $x-\psi$ plane, it is feasible to operate with a simple finite difference scheme, with constant intervals Δx, $\Delta\psi$. The differentials in the Laplacian (2) can be represented by the usual second order approximations:

$$\frac{\partial y}{\partial x} = \frac{y6 - y8}{2\Delta x} \qquad \frac{\partial y}{\partial y} = \frac{y5 - y7}{2\Delta\psi}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{y6 - 2y0 + y8}{\Delta x^2}$$

$$\frac{\partial^2 y}{\partial y^2} = \frac{y5 - 2y0 + y7}{\Delta\psi^2}$$

$$\frac{\partial^2 y}{\partial x \partial y} = \frac{(y_2 - y_1) - (y_3 - y_4)}{4\Delta x\Delta\psi}$$

the points are defined in figure 3. After replacing these differentials in (3) there results a large but sparse system of linear equations for the nodal value y0 in the x-ψ plane. A very simple and efficient method of solution to this system is the Successive Over Relaxation method (SOR), in conjunction with which a coefficient of overrelaxation w = 1.7 was found near optimal. In this solution, the field was subdivided into 90 to 180 intervals in the x direction and 10 to 20 intervals in the y direction.

3.2. The implementation of the surface boundary condition was conducted as follows: if dE = E – E(x) is the local difference between the true and computed values of the surface energy, then, from (4):

$$dE = \left[1 + \frac{V}{g}\frac{dV}{dy}\right] dy$$

the variation of the surface velocity $V = u_s\, \partial y_s/\partial x$ with a change of the surface depth dy is obtained from the second order finite difference approximations for the horizontal surface velocity us and surface slope dys:

$$us = \frac{2\Delta\psi}{3y_0 - 4y_1 + y_2} \qquad \frac{du_s}{dy} = -1.5\frac{u_s^2}{\Delta\psi}$$

hence the correction dy to ys, that restores the true value of the energy E to a section with surface energy E(x) is given by:

$$dy = \frac{E - E(x)}{1 - \frac{u_s^2}{gh}\left(1.5\frac{u_s h}{\Delta\psi}\left(1 + \frac{\partial y}{\partial x}\right)^2 - hy_s\right)}$$

where $h = y_s - y_b$.

This procedure, which changes only the values of y at the surface at each surface iteration is justified by noting that as each new boundary profile is defined there is a recalculation of the interior values of y at the nodal points. This procedure was found efficient and in general, it was possible to satisfy the constant pressure boundary condition in 5-10 iterations, provided that the initial estimate was not too different from the final profile. However, it was found that considerable care in the application of the surface correction algorithm (9) was necessary in the region where the Froude number was comprised between .3 to .8, as in this region the changes in y for a value of dE are maximal.

4. RESULTS

The large scale tests of Blau (1962) were used as control for the numerical method. These tests were conducted on a 1m wide canal, with a constant width weir of symmetric, parabolic form, of maximum height w = .322m and of a length of 1.60m. Water surface profiles and bottom pressure profiles were published for 6 unit discharges of q = .151 to .975 m2/s. Additional available information consisted of velocity and pressure profiles measured at the crest section.

To reproduce the Blau results by means of the potential flow method, the upstream section AF in figure 2 was located .40m upstream of the beginning of the weir and the downstream section .60m downstream of the crest. The theoretical water surface profiles and other results quoted below were determined by varying the value of the coefficient of discharge Cd in the discharge equation:

$$q = Cd\, h\sqrt{2g\, h} \qquad (10)$$

for a given q until a momentum balance was established between the upstream section AF and the crest section CC', taking into account the pressure distribution along the upstream face of the weir. This method resulted in calculated values of Cd that were slightly in excess of the experimental ones (figure 4). This was to be expected as the shear effects are neglected in the potential flow assumption, and the potential flow weir requires a slightly smaller head h than the real weir to pass the same flow. However, when the experimental values of Cd were used in the potential flow computations, the momentum difference amounted to no more than about .2% of the

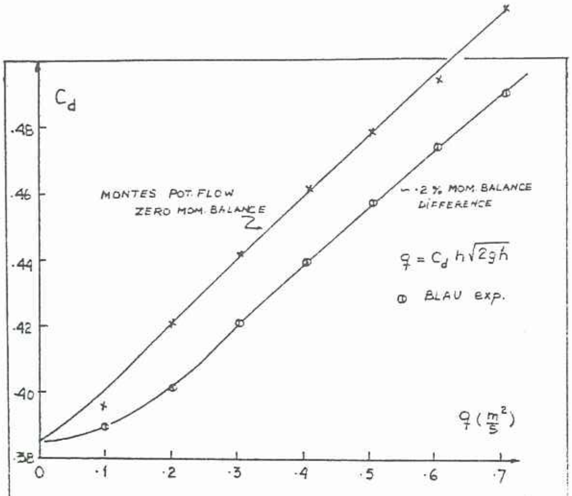

Figure 3 Coefficient of discharge of Blau's weir

original, upstream value. Confirming the hypothesis that on the accelerating part of the flow over the weir, the viscous effects are very small.

The comparison of the theoretical and experimental water surface profiles is shown in figure 5 for the cases of q = .17 and .51 m^2/s. The same figures detail also the bottom pressure profiles. Velocity distribution over the crest and the pressure profile are shown in figures 6 and 7.

5. DISCUSSION

The agreement of the calculated figures with the experiment is in most cases good, with correct trends and small percentual deviation of the computed results. This agreement provides support for the use of potential flow methods in the solution of highly curvilinear free surface problems. The x-y method here advocated is simpler in formulation and programming than other comparable methods, such as Strelkoff's integral equation method or Liggett and Liu's BEM. The main reason being that the y ordinate that defines the free surface is directly calculated from the solution of the Laplace equation and need not be obtained through an additional integration process as in other methods. As other researchers have found, the solution of the relatively large, but sparse matrices by SOR was almost trivially simple. The calculations for a single run need only 15-20 seconds on a 486-33 PC and require 5 to 10 free surface iterations to satisfy the zero pressure condition to within less than 1/1000 of the local depth.

Perhaps the only caveat in the application of this method together with free-surface searching algorithms of the type of equation (9) is the need for the first trial free surface profile to be a reasonable approximation of the experimental trend, or in its lack, one that satisfies approximately the energy constraints.

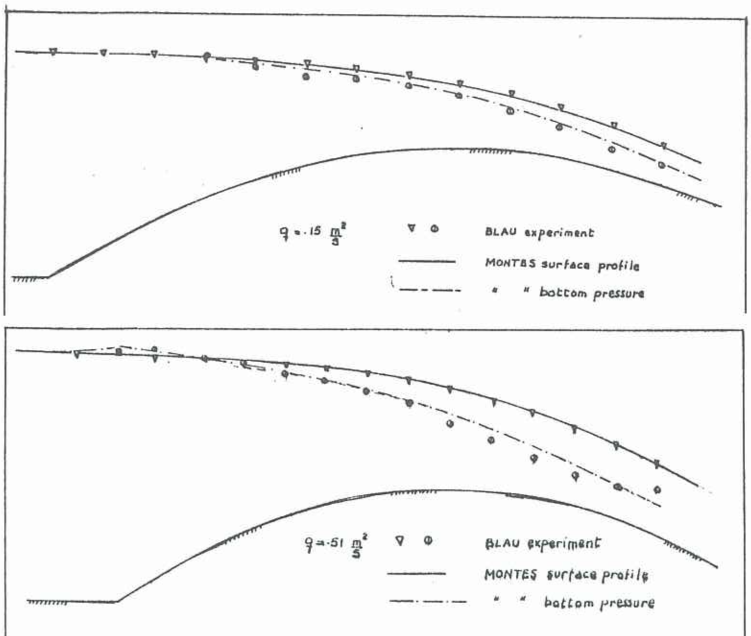

Figure 4 Comparison of calculated and experimental surface and bottom pressure profiles

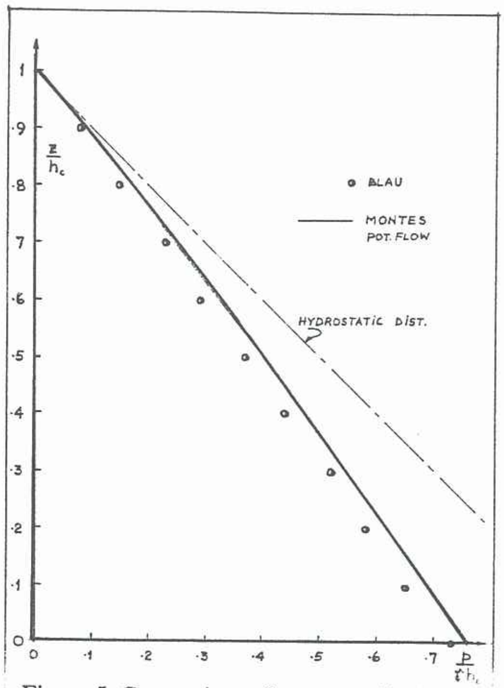

Figure 5 Comparison of pressure distribution on crest

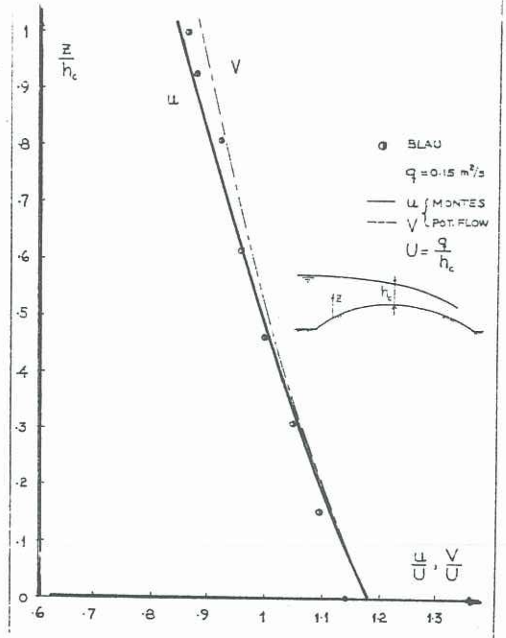

Figure 6 Comparison of velocity distribution on crest

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