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REGULARISATION OF POINT-VORTEX METHODS BY THE INCLUSION OF VISCOUS EFFECTS

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ABSTRACT

Point-vortex methods for modelling 2-D fluid flows commonly develop high-wavenumber instabilities which result in irregular vortex motion. Various schemes have been introduced to regularise the point-vortex equations, such as spectral filtering (Krasny, 1986a), and the use of 'vortex-blob' methods (e.g. Chorin and Bernard, 1973, Krasny, 1986b). These schemes are, however, somewhat ad hoc, and motivated simply by the need to suppress instabilities.

Here, a vortex method which models the effects of viscosity is used to calculate the evolution of a cylinder of fluid released in a uniform flow, a problem examined by Rottman et al. (1987). The solution remains smooth over the calculation interval for moderate Reynolds number ($Re = 1000$). The results of the calculation are contrasted with results gained using $Re = 10000$ (approaching the inviscid solution), with and without the spectral filtering technique introduced by Krasny, both of which develop instabilities over the same calculation time. Particular emphasis is placed on the differences in the evolution of the discrete spectra of the interfaces.

1. INTRODUCTION

Point-vortex methods are most useful when a 2-D flow problem involves concentrated regions of vorticity (such as a vortex sheet) embedded in large regions of irrotational fluid. The entire problem may then be solved by calculating the evolution of the vorticity distribution. Problems which can be modelled using a vortex sheet are those with a tangential velocity discontinuity across an interface separating two fluids. If the fluids are unbounded, and have the same density, a discretisation of the vortex sheet using N point vortices reduces the flow problem to the solution of 2N first order ODE's. If the fluids have different (constant) densities, Bernoulli's equation must also be solved to obtain the baroclinic evolution of the vortex-sheet strength (Baker et al. 1982).

The evolution of an inviscid vortex sheet is typically governed by a dispersion relation where normal mode growth rates increase without bound for increasing wavenumber. The discrete dispersion relation for such a sheet modelled using point vortices shows similar behaviour, with the maximum normal mode growth rate being that for the highest wavenumber. Since computers use only finite precision in calculations, any representation of an interface will have finite amplitudes at all wavenumbers of its spectrum (this amplitude being greater than or equal to the machine 'noise' level) and these amplitudes will grow according to the discrete dispersion relation, quickly leading to irregular vortex motion. This effect was investigated by Krasny (1986a), where the evolution of a periodic vortex sheet was used to model Kelvin-Helmholtz instability. A spectral filtering scheme where all spectral amplitudes below a fixed noise level were set to zero at each time step was introduced to overcome the problem. This filtering results in smoother solutions up to the time when all spectral amplitudes have grown past the noise level, after which the filtering has no effect and irregular vortex motion soon occurs.

Vortex-blob schemes, where the vortex sheet is represented by finite size vortex blobs (with a given vorticity distribution), rather than point vortices, have been used by several researchers (e.g. Chorin and Bernard, 1973, Krasny, 1986b, Rottman et al. 1987). The use of finite size vortex blobs has the effect of damping the growth of high-wavenumber amplitudes in the discrete dispersion relation for the evolution equations, and provides significant smoothing of solutions for large times. The use of arbitrary size vortex blobs is, however, motivated simply by the need to control high-wavenumber instability.

If viscosity is included in the problem formulation, then the discretisation of the vorticity distribution must satisfy the viscous vorticity transport equation. This can be achieved if vortex blobs which individually satisfy the equation are used to discretise the given distribution (see the review by Leonard, 1980). As would be expected, the inclusion of viscosity in this way effectively damps the high-wavenumber instability of the inviscid problem. This use of vortex blobs is more physically realistic than the use of arbitrary, constant size vortex blobs.

2. METHOD

We introduce the fluid vorticity ω, and the 2-D stream function Ψ, defined by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad \text{and} \qquad \mathbf{u} = (u, v) = (\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x})$$

where $\mathbf{u}$ is the fluid velocity. The governing equations of the fluid flow are then

$$\nabla^2 \Psi = -\omega \tag{1}$$

and

$$\frac{D\omega}{Dt} = \nu \nabla^2 \omega \tag{2}$$

where $\frac{D\omega}{Dt}$ is the material time derivative and ν is the kinematic viscosity.

The vorticity distribution is modelled by a sum of N vortex blobs

$$\omega(\mathbf{x}) = \sum_{i=1}^{N} \zeta(|\mathbf{x} - \mathbf{x}_i(t)|, \sigma(t))\gamma_i \tag{3}$$

where $\zeta(|\mathbf{x} - \mathbf{x}_i(t)|, \sigma(t))$ is the vorticity distribution function for the vortex blob centred at $\mathbf{x}_i$, $\sigma(t)$ is the vortex blob 'size' and γ_i is the effective far-field vorticity of the blob.

To satisfy equation (2), we use a Gaussian vorticity distribution for each vortex blob, with

$$\zeta(r, \sigma(t)) = \frac{1}{\pi[\sigma(t)]^2} \exp(-\frac{r^2}{[\sigma(t)]^2}) \tag{4}$$

and

$$[\sigma(t)]^2 = 4\nu t. \tag{5}$$

We can solve equation (1) using the appropriate Green's function for the flow geometry. For unbounded flow, this is

$$G_U(\mathbf{x} - \mathbf{x}') = \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}'|.$$

The velocity field at $\mathbf{x}$ can then be evaluated as

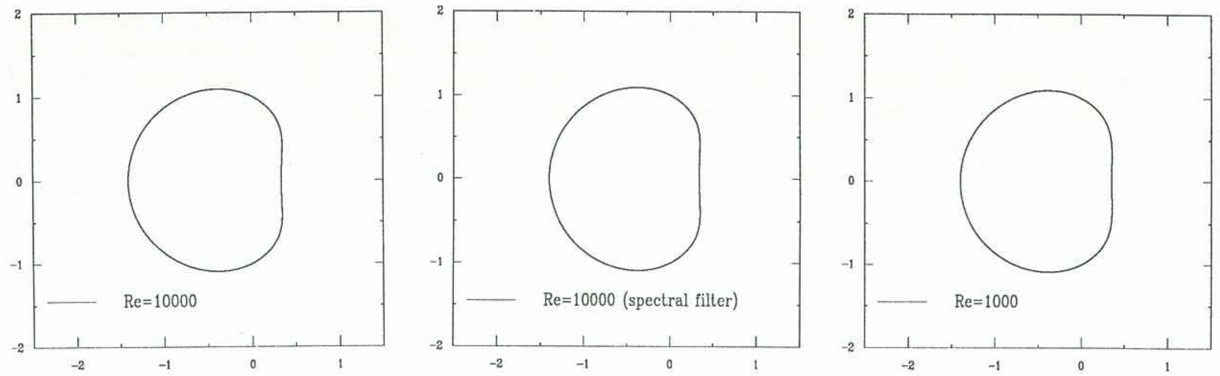

Figure 1(a). Interface shape at t=0.5.

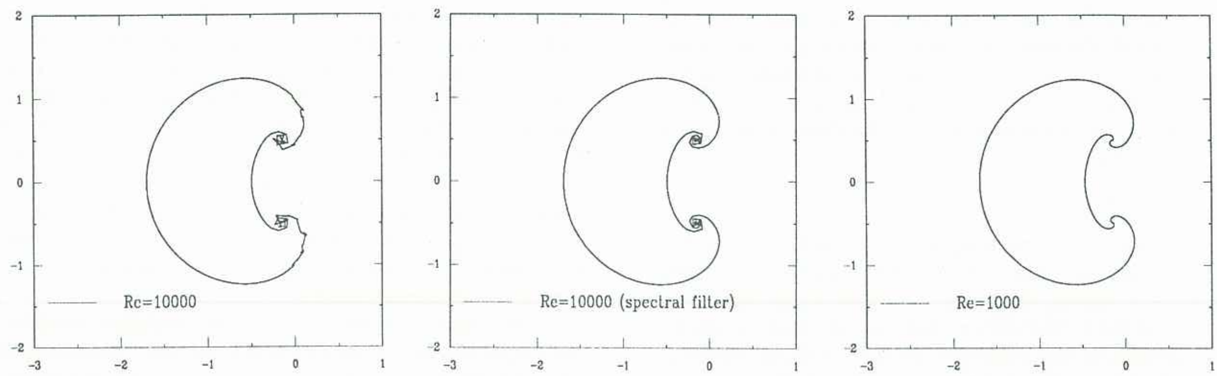

Figure 1(b). Interface shape at t=1.0.

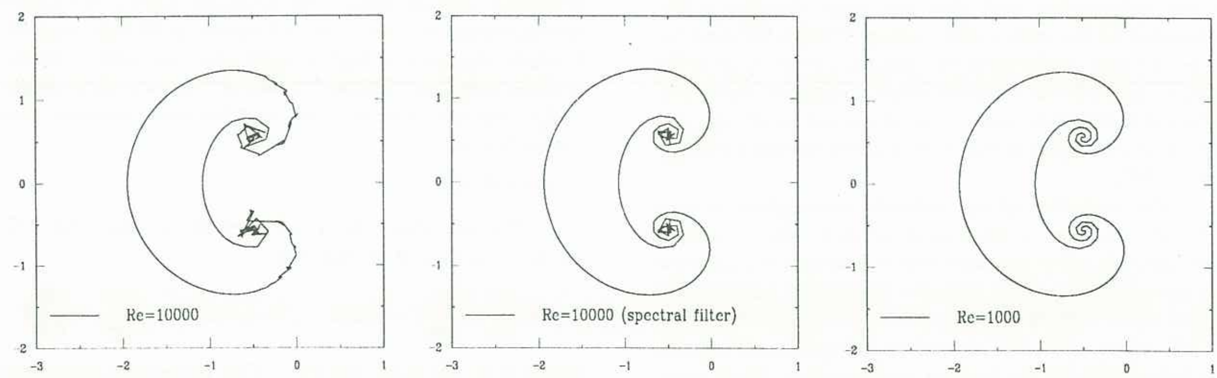

Figure 1(c). Interface shape at t=1.5.

$$\mathbf{u}(\mathbf{x}) = (u, v) = -\frac{1}{2\pi}\sum_{i=1}^{N}\gamma_i \frac{[1-\exp(-\frac{|\mathbf{x}-\mathbf{x}_i|^2}{\sigma^2})]}{|\mathbf{x}-\mathbf{x}_i|^2} \times ((y-y_i),(x_i-x)) \quad (6)$$

The advection of the vortex blobs is approximated by choosing the velocity at the centre of each blob to be the velocity at which the entire blob is advected, giving a set of 2N ODE's

$$\dot{\mathbf{x}}_j = (\dot{x}_j, \dot{y}_j) = -\frac{1}{2\pi}\sum_{i=1}^{N}\gamma_i \frac{[1-\exp(-\frac{|\mathbf{x}_j-\mathbf{x}_i|^2}{\sigma^2})]}{|\mathbf{x}_j-\mathbf{x}_i|^2} \times ((y_j-y_i),(x_i-x_j)) \quad (7)$$

to be solved at each time step.

The advection of the entire blob at one velocity preserves the blobs shape, whereas in reality, the blob would become distorted by any gradient in the velocity field. This approximation does introduce error to the advection of each vortex blob, on the order of σ^2 (Leonard, 1980).

Note that the velocity field (6) has no singular points (excepting $t = 0$), as would a velocity field produced by point vortices. The singular point-vortex equations are regained from (7) as $\nu \to 0$ and the $i = j$ terms are excluded. The regularisation of (6) when vortex blobs are used is the motivation behind ordinary vortex-blob methods. The method used in this paper could in fact be simply interpreted as an ordinary vortex-blob method where the strength of the regularisation used increases with time. This in itself would make the method presented here preferable to ordinary vortex-blob methods, since the error introduced to the calculation as the vortex blobs are advected is dependent on σ, as mentioned above.

3. RESULTS

The problem investigated is the evolution of a cylinder of fluid, initially at rest, which is released in a uniform crossflow, as was studied by Rottman et al. (1987). The cylinder may be represented as a circular vortex sheet, where the vortex sheet strength is given by the initial tangential velocity jump across the interface. For a cylinder with radius $R = 1.0$ in a uniform x crossflow with velocity $\mathbf{U} = (U, V) = (1.0, 0)$, the vortex sheet strength is

$$\omega(e) = -2\sin(e) \quad (8)$$

where e is an angular parameter $0 < e < 2\pi$. We introduce a Reynolds number $Re = \frac{2\pi RU}{\nu} = \frac{2\pi}{\nu}$ and use $N = 200$ vortex blobs to represent the interface. The spectral filtering (Krasny, 1986a) is performed by setting all spectral amplitudes of the interface position below $\epsilon = 10^{-14}$ to zero. The spectral amplitudes of the interface are calculated using Fast Fourier Transforms. Equation (7) is solved using a $4th$ order Runge-Kutta method with a time step of $\delta t = 0.005$ in a frame of reference moving with the uniform crossflow.

Figures 1(a), (b) and (c) (straight line interpolation between centres of vortex blobs) show three calculations of the interface at times $t = 0.5$, $t = 1.0$ and $t = 1.5$ respectively. The calculations were done for $Re = 10000$ with and without spectral filtering, and for $Re = 1000$. The $Re = 1000$ calculation remains smooth at all times, whereas the $Re = 10000$ calculations both develop irregularities, this effect being more severe for the calculation which is not spectrally filtered. The onset of instability at the highest wavenumber of the interface spectrum, which results in sawtooth irregularities on the interface, can be easily seen in figure 2, which shows the curvature κ of the interface versus e (the angular parameter),

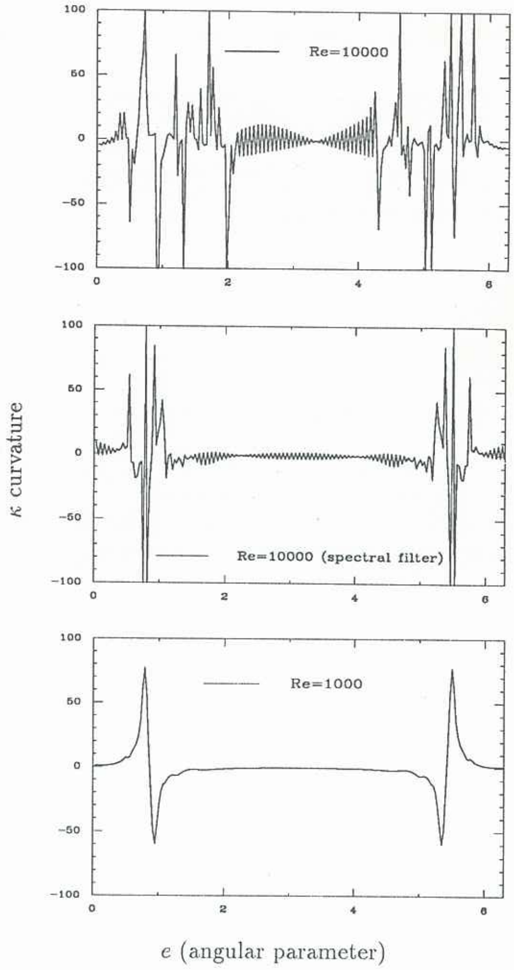

Figure 2. Interface curvature at t=1.0.

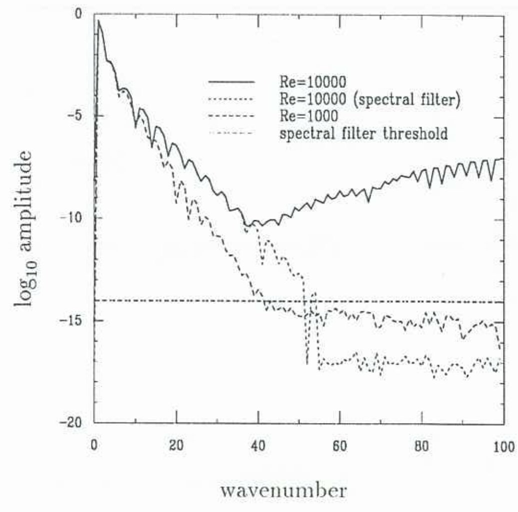

Figure 3(a). Interface spectrum at t=0.5.

where

$$\kappa = \frac{y_e x_{ee} - x_e y_{ee}}{(x_e^2 + y_e^2)^{\frac{3}{2}}}. \qquad (9)$$

The e derivatives were calculated spectrally using Fast Fourier Transforms. The interface develops very high curvatures at roll-up points, and in figure 2 curvatures with magnitudes greater than 100 have been truncated to ± 100 to retain legibility.

Figures 3(a), (b) and (c) show interface spectra at $t = 0.5$, $t = 1.0$ and $t = 1.5$ respectively, for the above calculations. The growth of high-wavenumber noise in the $Re = 10000$ case can be easily seen. The spectrum of the filtered $Re = 10000$ calculation becomes qualitatively similar to the unfiltered case after all wavenumbers have magnitudes above the noise level, and the solution becomes irregular. In the $Re = 1000$ case, however, high-wavenumber amplitudes are being continually damped, resulting in a smooth solution.

Figure 4 shows the increase of the arc-length of the interfaces in the above calculations with time. Arc-length s is calculated as

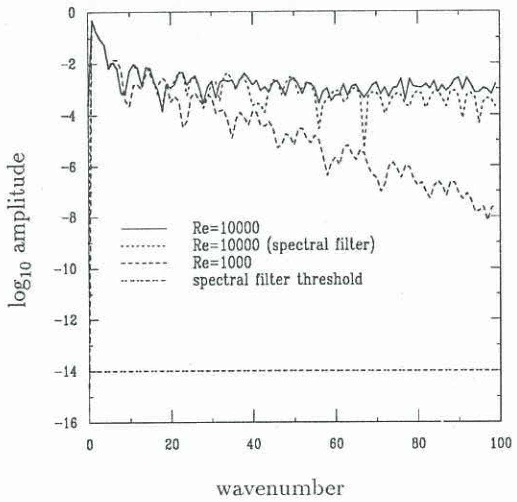

Figure 3(b). Interface spectrum at t=1.0.

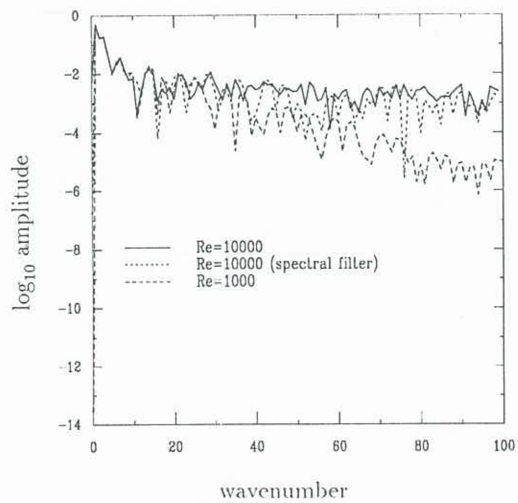

Figure 3(c). Interface spectrum at t=1.5.

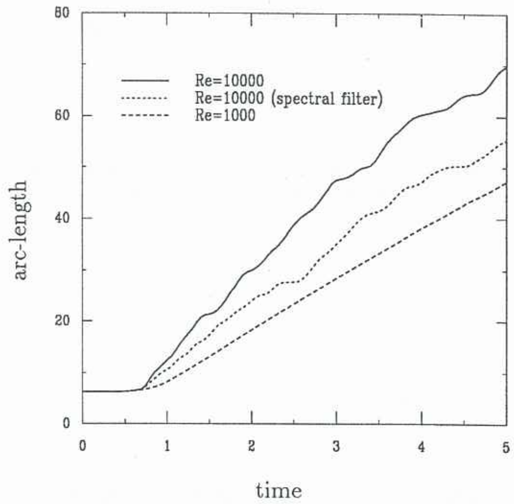

Figure 4. Arc-length increase with time.

$$s = \int_0^{2\pi} (x_e^2 + y_e^2)^{\frac{1}{2}} de. \qquad (10)$$

As would be expected, high-wavenumber irregularities lead to faster increase of arc-length with time.

4. CONCLUSION

The method presented is one motivated by the effects of viscosity and is more physically justifiable than arbitrary size vortex-blob methods. It is found to be successful in calculating smooth solutions for the evolution of a vortex sheet, due to the damping of high wavenumbers in the solution spectrum.

5. REFERENCES

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