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THE MEAN VORTICITY DOWNSTREAM OF A HYDRAULIC JUMP

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1. Introduction

In a recent publication, Yeh (1991) discusses the mechanism of vorticity generation in a steady-flow hydraulic jump. Three contributions are identified: Viscous shear at the interface between the two fluids, the baroclinic torque brought about by the static pressure gradient in the upper fluid, and the baroclinic torque brought about by the dynamic pressure gradient associated with a suitable velocity field in the upper fluid. The last of these is shown to dominate the other two and is proportional to the density ratio. It requires the vertical component of the pressure gradient in the upper fluid to be of opposite sign to that corresponding to the static gradient, so that a significant velocity field with prescribed features has to be present in the upper fluid. Since this velocity field will depend on whether the far-field upper fluid is at rest relative to the wave or relative to the upstream lower fluid, the vorticity generation would be different in these two cases. When the density ratio is very large, such as for an air/water interface, where it is approximately 1000, such a dependence on the motion of the tenuous upper fluid seems too sensitive. One might carry the argument to the extreme case of a mercury vapour/liquid interface, and ask whether the vapour motion would be able to influence the vorticity downstream of a hydraulic jump in the liquid. In that case the density ratio is $O(10^7)$, and it is hard to believe that the dimensionless vorticity would be 10000 times as large in the mercury case than in the water/air case at the same Froude number.

The present investigation was motivated by the dissatisfaction experienced by the author with this result. The fact that the momentum vector of the fluid entering the jump and the momentum vector of the fluid leaving it are not collinear, stimulated the application of the conservation of angular momentum to a control volume surrounding the jump. In this manner it was hoped that, just as in the classical derivation of the jump conditions, the omission of the conservation of mechanical energy would allow *unresolved* dissipative processes in the control volume to occur, and yet permit the jump conditions and the mean downstream vorticity to be determined.

The aim of this work is thus to consider a steady-flow hydraulic jump in a constant-density fluid on a horizontal, frictionless, solid surface, when the fluid has a density very much larger than that of the overlying fluid, so that the pressure at the free surface may be considered to be uniform, with a view to determining the mean vorticity downstream of the jump by the application of the conservation of angular momentum. This approach leaves the mechanism of vorticity generation unspecified.

2. The classical jump conditions.

The classical equations connecting the conditions upstream and downstream of a hydraulic jump are derived from a consideration of the conservation of mass and momentum in a control volume reaching to regions upstream and downstream where the flow is considered to be uniform. This derivation is repeated here as a form of introduction of the variables of the problem. Fig. 1 shows the control volume of the classical situation, with uniform velocity profiles upstream and downstream of the jump.

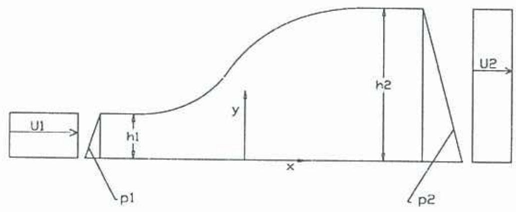

Fig. 1 Schematic sketch of hydraulic jump with control volume.

In terms of the quantities defined by Fig. 1, the conservation of mass across the jump is ensured, if

$$h_1U_1 = h_2U_2,$$

or

$$\frac{h_2}{h_1} = \frac{U_1}{U_2}. \tag{1}$$

Similarly, the conservation of momentum requires (in the absence of friction on the bottom) that

$$\frac{\rho g h_1{}^2}{2} + \rho U_1{}^2 h_1 = \frac{\rho g h_2{}^2}{2} + \rho U_2{}^2 h_2,$$

or, manipulating this by using (1),

$$\frac{U_1{}^2}{g\,h_1} = \frac{1}{2}\frac{h_2{}^2}{h_1{}^2}\left(1+\frac{h_1}{h_2}\right). \tag{2}$$

Now introduce the definitions of the Froude number F:

$$F = \frac{U_1^2}{g\,h_1}, \tag{3}$$

and the height ratio H:

$$H = \frac{h_2}{h_1}, \tag{4}$$

in order to rewrite equation (2) in the form

$$F = \frac{H^2}{2}\left(1 + \frac{1}{H}\right). \qquad (5)$$

It is useful to note here that, in the limit

$$F \to 1,\ H \to 1,$$

and in the limit

$$F \to \infty,\ H \to \sqrt{2F}.$$

3. Jump conditions with downstream vorticity

We now anticipate that the velocity profile downstream of the jump will be rotational and give it not only a mean velocity U_2, but in addition a mean vorticity ω. Thus the velocity distribution on the downstream side of the jump is now assumed to be

$$U = U_2 + \omega\left(\frac{h_2}{2} - y\right), \qquad (6)$$

where y is the distance from the horizontal solid bottom, measured vertically upward.

This change does not affect the mass balance, but the momentum balance has to be modified. It now requires that

$$\frac{g\,h_1}{U_1^2} + 2 = \frac{h_2^2}{h_1^2}\frac{g\,h_1}{U_1^2} + 2\frac{h_1}{h_2} + I,$$

where

$$I = 2\frac{h_1}{h_2}\int_0^1 \left[\frac{2\omega}{U_2}\left(\frac{h_2}{2} - y\right) + \frac{\omega^2}{U_2^2}\left(\frac{h_2}{2} - y\right)^2\right] d\left(\frac{y}{h_2}\right).$$

The first term does not contribute, and

$$I = 2\frac{h_1}{h_2}\left(\frac{\omega h_2}{U_2}\right)^2\left(\frac{1}{4} - \frac{1}{2} + \frac{1}{3}\right),$$

or

$$I = \frac{1}{6}\frac{\Omega^2}{H}, \qquad (7)$$

where

$$\Omega = \frac{\omega h_2}{U_2}. \qquad (8)$$

Substituting this back into the momentum balance and writing the new result in the dimensionless variables, we obtain the new relation

$$\frac{H^2}{2F}\left(1 + \frac{1}{H}\right) = 1 - \frac{\Omega^2}{12(H-1)}. \qquad (9)$$

The difference between the previous result, equation (5) and this is that a new term in Ω appears on the right. Ω is of course not known until a new condition is applied.

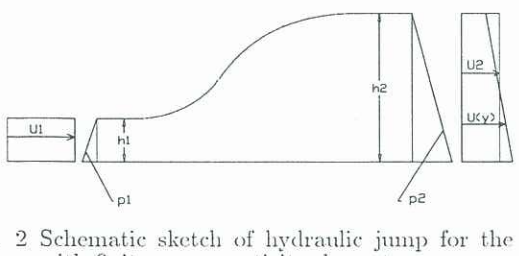

Fig. 2 Schematic sketch of hydraulic jump for the case with finite mean vorticity downstream

The appropriate new condition is the conservation of angular momentum, which demands that the torque applied to the control volume by external forces be equal to the rate of change of the angular momentum of the fluid contained in it. The procedure is considerably more complex than in the conservation of linear momentum, because in the latter, only the streamwise component needs to be considered, so that the force on the bottom (in the absence of friction) and body forces are not needed. In order to simplify the process, we consider the case naively first, by neglecting the vertical inertial forces, *i.e.* by assuming that the pressure distribution on the bottom exactly balances the weight of the fluid. This eliminates the vertical forces from the consideration again.

Taking moments about a point in the bottom surface (to aid the thinking, one might choose the centre of pressure, so that, with the above assumption, the weight and bottom pressure force do not contribute) the angular momentum balance yields

$$\frac{g{h_1}^3}{6} + \frac{U_1^2 h_1^2}{2} = \frac{g{h_2}^3}{6} + \frac{U_2^2 h_2^2}{2}$$

$$+ \int_0^{h_2} \left(2\,U_2\,(U - U_2) + (U - U_2)^2\right) y\,dy.$$

It is important to realize that the point about which one takes moments does not affect the result. This is because the linear momentum is balanced separately, and the terms introduced by a change in the fulcrum are zero by virtue of the linear momentum balance. Rewriting the above equation with a linear velocity profile (equation (6)), using the continuity equation (1), and performing the integration, a quadratic equation results:

$$\frac{1 - H^3}{F} = \Omega\left(\frac{\Omega}{4} - 1\right). \qquad (10)$$

The only physically interesting solution of this equation is

$$\Omega = 2\left[1 - \sqrt{\frac{1 - H^3}{F} + 1}\right]. \qquad (11)$$

To check this result, consider what it does in the limiting cases of $F \to 1$ and $F \to \infty$. As $F \to 1$, $H \to F$, so that $\Omega \to 0$, which is intuitively appealing. However, at $F \approx 1.3$ the radicand changes sign, so that, for greater F, equation (11) gives a physically meaningless result.

4. The torque from the bottom pressure

Something must therefore be wrong in the derivation of (11). It is necessary to include the torque exerted on the fluid by the vertical forces in the conservation of angular momentum. If the vertical component of the inertial forces were zero, the bottom pressure would exactly balance the weight of the fluid, so that no net torque would be acting on the fluid because of the vertical force components. In the left half of the control volume there is a mean concave-up streamline curvature. To provide this curvature a transverse pressure gradient is required. Since the pressure at the free surface is independent of streamwise distance (assuming the density of the overlying fluid to be negligible) the pressure on the bottom must exceed the static pressure corresponding to the height of liquid above it. The opposite situation occurs in the downstream half of the control volume, where the mean streamline curvature is convex up, so that the bottom pressure is lower than it would be without this inertial force.

The additional bottom pressure distribution brought about by the vertical acceleration of the fluid is thus antisymmetric, and will exert a clockwise torque on the fluid. This also has the required feature that it disappears at $F \to 1$, since the mean streamline curvature disappears, and increases as F increases. Unfortunately, it is not possible to obtain it without some further assumptions. Let this torque be t per unit lateral distance and introduce the dimensionless torque

$$T = \frac{t}{\rho U_1^2 h_1^2}.$$

In order to study it in some detail, consider the differential form of the continuity and vertical momentum equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g,$$

where the symbols have their usual meaning. By using the continuity equation to replace $\frac{\partial v}{\partial y}$ in the momentum equation, and replacing p with the excess pressure p' over the static pressure according to

$$p' = p - \rho g(h - y),$$

the momentum equation becomes

$$-\frac{1}{\rho}\frac{\partial p'}{\partial y} = u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x} = u^2\frac{\partial}{\partial x}\left(\frac{v}{u}\right). \quad (12)$$

At the free surface,

$$v(x,h) = u(x,h)\,\frac{dh}{dx}.$$

Assume that

$$u\,h(x) = U_1\,h_1,$$

independent of y, and

$$v = v(x,h)\,\frac{y}{h} = U_1\,h_1\,\frac{dh}{dx}\frac{y}{h^2},$$

a linear profile satisfying the bottom condition $v(x,0) = 0$. Substituting these in equation (12), we obtain

$$-\frac{1}{\rho}\frac{\partial p'}{\partial y} = \frac{U_1^2\,h_1^2\,y}{h^2}\frac{d}{dx}\left(\frac{1}{h}\frac{dh}{dx}\right).$$

This may be integrated over y from 0 to h to give the excess bottom pressure:

$$2p_b' = -\rho\,U_1^2\,h_1^2\,\frac{d}{dx}\left(\frac{1}{h}\frac{dh}{dx}\right).$$

Consequently the clockwise torque per unit transverse distance exerted by the excess bottom pressure on the fluid is

$$T = \frac{t}{\rho\,U_1^2\,h_1^2} = -\frac{1}{2}\int_{-\infty}^{\infty} x\,\frac{d}{dx}\left(\frac{1}{h}\frac{dh}{dx}\right)dx. \quad (13)$$

In order to show how the excess pressure behaves, Fig. 3 shows this as a function of dimensionless distance for the case of a hyperbolic-tangent wave shape.

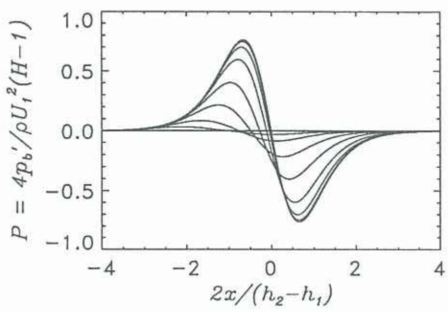

Fig. 3 The behaviour of the excess pressure (scaled by $(H-1)/2$) for a hyperbolic-tangent wave shape and values of $H-1 = 0.01$, 0.03, 0.1, 0.3, 1, 3, 10, and 30. In this representation the largest amplitude corresponds to the smallest value of H.

However, it is in fact not necessary to assume a particular wave shape, because (13) may be integrated by parts to give

$$T = -\frac{1}{2}\left[\frac{x}{h}\frac{dh}{dx} - \ln h\right]_{-\infty}^{\infty}. \quad (14)$$

For all wave shapes of interest here,

$$\frac{x}{h}\frac{dh}{dx} \to 0 \quad \text{as} \quad x \to \pm\infty,$$

and

$$T = \frac{1}{2}\ln\frac{h_2}{h_1} = \frac{1}{2}\ln H. \quad (15)$$

To the approximation made here, the torque from the excess bottom pressure is thus independent of the detail of the wave shape. This may be expected to be correct for small values of H, but care needs to be exercised at larger H. This is because the assumptions made earlier about the forms of $u(x,y)$ and $v(x,y)$ break down as $H \to \infty$.

One cause of this failure is evident from a consideration of the situation when the spill of the wave forms the roller that is observed at large H in a hydraulic jump. Clearly, the assumptions of uniform u over y and linear v over y fail in this case.

It is important to check that the excess bottom pressure does not produce a net force. This may easily be done by integrating the pressure distribution over x. The result is

$$\int_{-\infty}^{\infty} -\frac{2p_b'}{\rho\,U_1^2\,h_1^2}\,dx = \left[\frac{1}{h}\frac{dh}{dx}\right]_{-\infty}^{\infty},$$

which is zero.

5. The effect of the bottom pressure torque

Substituting the term introduced by t into the equation for angular momentum balance (10) as a correction, we obtain

$$\frac{1-H^3}{F} = \Omega\left(\frac{\Omega}{4} - 1\right) + 6T. \quad (16)$$

Solving for Ω,

$$\Omega = 2\left[1 - \sqrt{1 - \frac{H^3-1}{F} + 6\,T}\right],$$

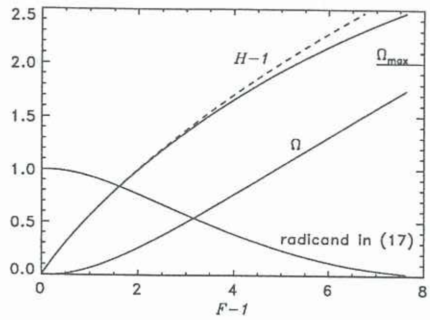

Fig. 4 The wave height in the form $H-1$ and the downstream vorticity Ω as functions of the Froude number $(F-1)$ as given by simultaneous solution of (9) and (17).

or

$$\Omega = 2\left[1 - \sqrt{1 - \frac{H^3-1}{F} + 3\ln H}\right]. \quad (17)$$

$\Omega(F)$ and $H(F)$ may now be obtained by solving (9) and (17) simultaneously. The result is plotted in Fig. 4.

The behaviour of Ω in the limits $F \to \infty$ and $F \to 1$ contain the most interesting features of our results. At $F-1 \to 0$ series solutions may easily be obtained for $F-1$ and Ω in increasing powers of $\epsilon = H-1$. The result is

$$\Omega = \frac{3}{4}\epsilon^3 - \frac{9}{8}\epsilon^4 + \frac{107}{80}\epsilon^5 + O(\epsilon^6), \quad (18)$$

and

$$F-1 = \frac{3}{2}\epsilon + \frac{1}{2}\epsilon^2 + \frac{3}{64}\epsilon^5 + O(\epsilon^6). \quad (19)$$

Thus the vorticity increases from zero as

$$\Omega = \frac{2}{9}(F-1)^3 + O(F-1)^4.$$

This is reminiscent of the manner in which dissipative effects behave in the analogous situation of a shock wave in a compressible fluid. The square of the Mach number M^2 corresponds to the Froude number F in the analogy and dissipative effects manifest themselves in the form of an entropy increase in the shock wave. This entropy change increases with the cube of M^2-1, just as the manifestation of dissipative effects in the hydraulic jump (namely vorticity) increases as the cube of $F-1$ here.

At the other extreme, the present solutions are not able to make a satisfactory prediction, because the approximations leading to the expression for the torque from the bottom pressure break down. However, it is possible to make an intelligent guess about the behaviour of the radicand of equation (17). The trend of this radicand is to decrease towards zero at $F \approx 9.5$ (see Fig. 4). It must not change sign, however, if the solution is to remain real. We expect, therefore, that the physical case corresponds to the radicand approaching zero smoothly as $F \to \infty$. If this is correct, Ω will approach the value 2 asymptotically. This is also the maximum value that can be reached, because it corresponds to $U(h_2) = 0$. Larger values of Ω imply negative $U(h_2)$, which does not make sense, because it corresponds to the downstream fluid overtaking the wave.

The value of Ω at $F \to \infty$ may be approximately assessed from the experimental evidence that a surfboard or other floating object is carried along almost at the same speed with a large broken wave even if it is just downstream of the wave. This indicates that, provided that the downstream velocity profile is approximately linear in y, the value of Ω at large F is less than, but nearly equal to 2. This lends additional support to our guess.

6. Conclusions

The exercise of applying the conservation of angular momentum to a control volume around a hydraulic jump with no friction on the horizontal bottom surface, yielded the following interesting results.

1. A solution was found for the mean vorticity downstream of the hydraulic jump which increases as the cube of $F-1$ as the Froude number F increases from 1, and tends to a constant dimensionless value (=2) as $F \to \infty$. The presence of the downstream vorticity reduces the height ratio slightly at large Froude number.
2. The vorticity is limited by the torque exerted on the control volume by the excess bottom pressure that arises from the vertical acceleration of the liquid. A closed-form solution was obtained for this torque which is independent of the wave shape and depends only on the height ratio.
3. While this argument does not provide the mechanism by which the vorticity is generated, it can give the rate of production by requiring that angular momentum is produced at a rate equal to the externally applied torque.

Vorticity generation is always related to dissipative processes, and in the case of the hydraulic jump, it is clearly connected with the fact that mechanical energy is not conserved in the jump. If it is assumed that the flow is steady and frictionless, as has been done here, no mechanism (other than the *steady-flow* baroclinic mechanism proposed by Yeh) is therefore provided by which vorticity may be generated. By invoking only the conservation of mass, momentum and angular momentum, and not the conservation of energy, it is possible, however, to obtain a solution to the problem, just as one does in the classical derivation of equation (5) *without* resolving the detailed mechanism of vorticity generation. Since the most important dissipative processes involved in the hydraulic jump are connected with the unsteadiness of the free surface, it may be concluded that the baroclinic torque associated with nonuniformly accelerated free surfaces is primarily responsible for the detailed vorticity generation.

7. Reference

Yeh H. H. 1991 Vorticity generation mechanisms in bores, *Proc. Roy. Soc. London*, *A*, **432** 215-231.