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A PARTICLE METHOD FOR THE SOLUTION OF A ONE-DIMENSIONAL NON-LINEAR EQUATION

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ABSTRACT: We study a modified particle method for solving the one dimensional nonlinear equation $u_t + f_x(u) = 0$. It involves creating an initial particle path and an initial approximation $\tilde{u}_0$ with error $\|u_0 - \tilde{u}_0\|_1 = O(N^{-1})$, where N is the number of particles. We compare the results of numerical experiments with time step $\Delta t = O(\sqrt{1/N})$ with the exact solutions for several test problems.

1. INTRODUCTION

Particle methods have been used to treat a broad class of problems. Puckett (1989) used a random particle method for the Kolmogorov equation. In fluid mechanics, these methods are called vortex methods and have been used for solving incompressible and viscous flows (Chorin, 1973 and Chorin and Marsede, 1979). In such methods particles represent point concentrations of some derivative of the solution.

We study a particle method for approximating the solution of one dimensional initial-value problems

$$\begin{aligned} &u_t + f_x(u) = 0 && \text{in } R \times R^+ \\ &u(x,0) = u_0(x) && x \in R \end{aligned} \tag{1}$$

This equation is a conservation law and arises in the study of nonlinear wave phenomena, when dissipation effects, such as viscosity, are neglected. When $f''(u) > 0$ and $u_0'(x) < 0$, the above equation (1) is not classically solvable even if $f(u)$ and $u_0(x)$ are analytic, and for some initial data the weak solution of this Cauchy problem loses uniqueness. It is our goal to develop a numerical method involving these equations which can approximate the physical solution.

In our method, the approximate initial data $\tilde{u}^0$ is a step function approximation to u_0 and is required to be monotonic (assume that u_0 is continuously differentiable on R and $u_0 \in L^1(R) \cap L^\infty(R)$) and the error for initial approximation is $O(N^{-1})$. The shock is treated by altering the speed of propagation for each particle which has been obtained from the "jump condition" satisfying the entropy condition. In the expansion case, the approximate rarefaction-wave solution is obtained by utilising the nonlinear character, initially creating a small pseudo-compression area between the exact wavefront and the approximate wavefront.

Here, we compute solutions of Burger's equation with different initial values. The solutions reveal that the shock wave or the rarefaction wave are determined by the initial data, not by "further events". An estimate of the error is given.

2. THE PARTICLE METHOD

In this section we describe the numerical approximation to weak solutions of the scalar Cauchy problem. Note that the equations (1) can be written as

$$\begin{aligned} &u_t + f'(u)u_x = 0 && \text{in } R \times R^+ \\ &u(x,0) = u_0(x) && x \in R \end{aligned} \tag{2}$$

where $u_0(x)$ is assumed to be a function of bounded total variation. It is well known that weak solutions of this problem are not unique. The total differential du is $u_0(x)$. If x and t are constrained to lie on a curve L, then at any point P on L we have

$$\frac{du}{ds} = u_t \frac{dt}{ds} + u_x \frac{dx}{ds} \tag{3}$$

where now dx/ds is the gradient of curve L at point P. Comparing (2) and (3), the equation (2) can be interpreted as the ordinary differential system

$$\frac{du}{ds} = 0 \tag{4}$$

$$\frac{dx}{ds} = f'(u) \tag{5}$$

$$\left.\frac{dt}{ds}\right|_{s=t} = 1 \tag{6}$$

Equations (4), (5) and (6) show that the characteristics of equation (2) are determined by three ordinary differential equations. It is easy to see that the characteristics intersect in the (x,t) plane. Since the slope of the characteristic is $1/f'(u)$, the characteristics have slope determined by their values at t=0, ie. $u_0(x)$. If there are points $x_1 < x_2$ with

$$m_1 = \frac{1}{f'(u_0(x_1))} < \frac{1}{f'(u_0(x_2))} = m_2$$

then the characteristics starting at $(x_1,0)$ and $(x_2,0)$ will cross in t>0. Thus at P, the solution must be discontinuous, and we can see analytically that discontinuities must form if u_0' is negative at some point. Assume that $f'' > 0$, since u is constant along the characteristic, then u must implicitly be given by

$$u(x,t) = u_0(x - tf'(u(x,t)))$$

So if u_0 is a differentiable function, then we can use the implicit function theorem to solve this equation for u. Provided that t is sufficiently small, we have

$$u_t = -\frac{f'(u)u_0'}{1 + u_0' f''(u)t}, \quad u_x = \frac{u_0'}{1 + u_0' f''(u)t}$$

It shows that if $u_0' < 0$ at some point, both u_t and u_x become unbounded when $1 + u_0' f''(u)t$ tends to zero. This implies that we cannot obtain a globally defined solution, and this conclusion is independent of the smoothness properties of u_0 and f. The phenomenon is a purely nonlinear one. An additional principle is needed to select the unique physical solution, that is given by the validly of an entropy inequality (Smoller, 1988). A particle method for solving the nonlinear equation (2) involves the approximate solution of equations (4)-(6). The processes are combined by moving the particles along their characteristic forwards with time, and then using the particles to represent the point concentration of some derivative of the solution at each particle position at that time. Thus, the velocity field can be obtained from these particles.

If $f(u)=(u^2/2)$, (1) becomes the inviscid Burger's equation

$$\begin{cases} u_t + uu_x = 0 & t>0 \\ u(x,0)=u_0(x) \end{cases} \tag{7}$$

which has the property that the only C^1 functions which satisfy it in $t>0$ are those which are monotonically nondecreasing in x for each fixed $t>0$. So, it is necessary to take a monotonic restriction initial data at each time step. This requires that the initial approximation u^0 is monotonic. We now assume initial data u^0 satisfies $u^0 \in C^1(R)$, $u'^0 < 0$. Let w_j^n denote the strength or weight for each particle, x_j^n denote the position of the jth particle at time $n\Delta t$, $\tilde{u}(x,t)$ denote an approximation to the solution $u(x,t)$ of equation (7), and $\tilde{u}_j^n = \tilde{u}^n(x_j^n)$ denote the value of $\tilde{u}^n$ at the jth particle position at time $n\Delta t$. The initial particle position x_j^0 is generated by taking the inverse of u^0. Then if $u^0(x)=g(x)$,

$$x_j^0 = \begin{cases} g^{-1}(1-\frac{j}{N}) & j=1,\cdots N \\ g^{-1}(1/2N) & j=N \end{cases} \tag{8}$$

Thus, the initial particle position is a monotonic sequence. The strength for each particle is given by

$$w_j^0 = \frac{|u(x_j^0,0)|_{\max}}{N} \tag{9}$$

Here for convenience, let $|u(x_j^0,0)|_{\max}=1$.

Now, we approximate u^0 using N particles, each with fixed strength w_j^0 , so that

$$\tilde{u}_j^0 = \sum_{i=1}^{N} H(x_i^0 - x_j^0) w_i^0 \tag{10}$$

where $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x<0 \end{cases} \tag{11}$$

Next we make the approximate solution to satisfy the monotonic restriction at each time step. Notice that if the strengths are initially chosen as $w_j^0 = 1/N$, the strength of each particle satisfies

$$\begin{cases} 0 < w_j^0 \le 1 \\ \sum_{j=1}^{N} w_j^0 = 1 \end{cases} \tag{12}$$

Since x_j^0 is an ordered monotonic sequence, ie. $x_i^0 \le x_2^0 \le \cdots \le x_N^0$, it implies that $0 \le u_j^0 \le 1$ (13)

and we obtain the initial approximation $\tilde{u}_j^0 = 1$ only if $j=1$. The approximation of u^0 obtained by N particles will have order

$$\left\| u^0 - \tilde{u}^0 \right\|_1 = O(1/N) \tag{14}$$

Now, we will see the initial distribution transported conservatively by particles in the flow. This involves the solution of the ordinary differential equations (4)-(6). First, we solve the characteristic equation (4) to update particle positions

$$\begin{cases} \dot{x} = u \\ x(0) = x_j^0 \end{cases} \tag{15}$$

Using Euler's method, we have

$$x_j^{n+1} = x_j^n + \Delta t \sum_{i=1}^{N} H(x_i^n - x_j^n) w_i^n \tag{16}$$

Thus, we obtain the updated positions for each particles at time $(n+1)\Delta t$, where the time step is chosen as $\Delta t = \sqrt{1/N}$. To obtain the velocity for each update particle position, we solve the ordinary differential equation

$$\dot{u} = 0 \tag{17a}$$

Using the upwinding finite difference scheme, we obtain

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = 0 \tag{17b}$$

Equation (17b) implies the strength of each particle is conserved in each time step, that is

$$w_j^{n+1} = w_j^n \tag{17c}$$

It is easy to show that $\sum_j w_j^n = 1$ implies $\sum_j w_j^{n+1} = 1$, and that for time $0 \le t^k \le t^{k+1}$, since $\tilde{u}_j^n$ is a particle solution, also $\tilde{u}_j^{n+1}$ is a particle solution. The solution at time $t^k = (n+1)\Delta t$ is given by the combination with all the particles,

$$\tilde{u}^{n+1}(x,t) = \sum_{j=1}^{N} H(x - x_j^{n+1}(t)) w_j^{n+1}(t) \tag{18}$$

Since we have exactly obtained the particle positions, and noted equations (17) say that the strength of each particle is conserved along the solved characteristic line, we have the following relations

$$\left| u^0 - \tilde{u}^0 \right| = \left| u^0 - \sum H(x - x_j^0(t)) w_j^0(t) \right| \le O(1/N) \tag{19a}$$

and $$\min_{x\in R} u^0 \le \tilde{u}_j^0 \le \max_{x \in R} u^0 \tag{19b}$$

At time $t>0$, the error increases and has a maximum error before the shock like $O(\sqrt{1/N})$. Now consider the case when the time $t \ge t^{k+1}$. The characteristic theory gives a multi-valued solution. It says the shock has been formed and so if we continue to solve, this will induce the solution to "blow up" at the shock front. In this case, we can consider simply altering the speed of propagation for each particle. Denoting this speed by s, the "jump condition" satisfies the entropy condition,

$$s_j^{t^{k+1}} = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{u_l + u_r}{2} \tag{20}$$

We take the left limiting value u_l as $\tilde{u}_1^{t^k}$, the right limiting value u_r as $\tilde{u}_N^{t^k}$, and in progress, once $x_1^{t^{k+1}} \ge x_2^{t^{k+1}}$, then go back one time step to recalculate the particle position with the speed $s_j^{t^{k+1}}$ which is obtained from (20),

$$x_j^{t^{k+1}} = x_j^{t^k} + \Delta t s_j^{t^{k+1}} \tag{21}$$

Thus, the solution for $t \ge t^{k+1}$ is given by

$$u_j^{t^{k+1}} = \sum_{i=1}^{N} H(x_i^{t^{k+1}} - x_j^{t^{k+1}}) w_i^{t^{k+1}} \tag{22}$$

Notice that the strength of each particle does not alter at the shock front, and thus, once the shock is formed, the max-error for $t \ge t^{k+1}$ can be measured by calculating the area error

$$s = \left| \int_{x_1}^{x_N} (u - \tilde{u}) dx \right| \approx O(1/2\sqrt{N}) \tag{23}$$

That is, the approximate solution $\tilde{u}_j^{n+1}$ will have order $O(1/2\sqrt{N})$ for all $t \ge t^{k+1}$.

3. THE RAREFACTION WAVE

We now consider initial data which is piecewise constant, such as

$$u(x,0) = \begin{cases} u_l & x < x_0 \\ u_r & x \ge x_0 \end{cases} \tag{24}$$

In this case, the density decreases continuously from u_l to u_r as time increases, and the contact discontinuity is due to the original discontinuity in the data. As time increases, the solution will form as the "rarefaction-wave". To solve this problem, we utilise the inherent character of the nonlinear equation that the solution converges to the rarefaction wave.

First, we create a small region which has the compression property relative the approximation wave front locally and the expansion property globally. This small region can be specified as $[|x_0|,|x_0|+\frac{1}{N}]$. Then, place N particles in this interval, take the strength of each particle for u_l is $w_{j,l}=u_l/N$ and the strength of each particle for u_r is $w_{j,r}=u_r/N$. Let $\tilde{u}_l$ and $\tilde{u}_r$ be the approximations to the contact discontinuity u_l and u_r

$$\begin{cases} \tilde{u}_j^{0,l}=\sum_{i=1}^{N} H(x_i^{0,l}-x_j^{0,l})w_i^{0,l} \\ \tilde{u}_j^{0,r}=\sum_{i=1}^{N} H(x_i^{0,r}-x_j^{0,r})w_i^{0,r} \end{cases} \tag{25a}$$

and

$$\begin{cases} \tilde{u}_j^{n+1,l}=\sum_{i=1}^{N} H(x_i^{n+1,l}-x_j^{n+1,l})w_i^{n+1,l} \\ \tilde{u}_j^{n+1,r}=\sum_{i=1}^{N} H(x_i^{n+1,r}-x_j^{n+1,r})w_i^{n+1,r} \end{cases} \tag{25b}$$

Let the particles travel at the approximation speed, then we obtain the position for $t=(n+1)\Delta t$

$$\begin{cases} x_j^{n+1,l}=x_j^{n,l}+\Delta t\tilde{u}_j^{n,l} \\ x_j^{n+1,r}=x_j^{n,r}+\Delta t\tilde{u}_j^{n,r} \end{cases} \tag{26}$$

The interesting thing in this particular problem is that, after choosing our initial approximation to initial data, the inherent nonlinear character of the equation causes the error to decrease with time and the approximate solution to converge to the exact solution as time tends to infinity

$$\lim_{n\to\infty} |u-\tilde{u}^n|=0$$

We have found that as time increases the local error decreases like

$$|err^{n+1}|_{\max}\approx O(\frac{\Delta t}{n}) \tag{27}$$

and the maximum error at each time step is also like $O(\frac{\Delta t}{n})$.

The total error of area between u and $\tilde{u}$ may expressed as

$$|S-\tilde{S}|=\left|\int_{[x_0,x_0+\frac{1}{N}]}|u-\tilde{u}|dx\right| \tag{28}$$

and the area change with time may be written as

$$\begin{cases} \frac{dS}{dt}=\mathrm{V} \\ \frac{d\tilde{S}}{dt}=\tilde{\mathrm{V}} \end{cases} \tag{29}$$

where, V and $\tilde{\mathrm{V}}$ are the expansion speeds in the u and $\tilde{u}$ covered areas respectively, and hence relate to u and $\tilde{u}$.

In the small "pseudo-compression" region $[|x_0|,|x_0|+\frac{1}{N}]$, the density suffers pseudo-compression relative the approximation wavefront only locally. We use the term "pseudo-compression" to indicate that effect looks like a compression wave, because the propagation speed in the small region is faster than the approximation wave, and after some time, it should catch the approximation wave, but no such shock will be formed because it is not a real compression wave. In fact, the density behind the real wave must decrease with time, and the real wave front must form as the expansion wave. In the small region, the expansion speed decreases with time while the fast propagation speed (compared with the approximation speed) moves the wavefront to pursue the approximation wave. So, the small area decreases in time. Once the two wave fronts overlap at some time $t=t_n$, they will have the same expansion speed and the same wavefront expansion. For $t>t_n$ and $u\geq\tilde{u}$, then $\mathrm{V}\geq\tilde{\mathrm{V}}$, $\mathrm{V}^{n+1}\leq\mathrm{V}^n$, and $\tilde{\mathrm{V}}^{n+1}\leq\tilde{\mathrm{V}}^n$, which imply

$$\left|S^{n+1}-\tilde{S}^{n+1}\right|\leq\left|S^n-\tilde{S}^n\right| \tag{30}$$

and

$$\left|\frac{e^{n+1}}{e^n}\right|\leq 1 \tag{31}$$

Expression (31) implies that the particle solution will weakly approximate the exact solution, and the convergent speed relates to its accuracy at each time step. Notice that the speed of convergence is related to the small pseudo-compression area. Since

$$\left|(x_0+\frac{1}{N})-x_0\right|\leq\left|(x_0+\frac{1}{2N})-x_0\right|\leq\cdots\leq\left|(x_0+\frac{1}{jN})-x_0\right| \tag{32}$$

we have an alternative to placing a large number of particles in the initial approximation region. We have found that if we take $j\geq2$, the approximate solution will converge to the exact solution at a rate j times the original rate. This implies that the error will decrease at the rate $O(\Delta t/jn)$.

4. NUMERICAL RESULTS

We present several test problems by taking different initial values for the inviscid Burger's equation (2). In case (a), we set the particle number N=1000, and a time step $\Delta t=1/32$. In case (b) and (c), we use only 100 particles and a time step Δt=0.1.

(a) Consider continuous initial data with $u_0'(x)<0$.

$$u(x,0)=\begin{cases} 1 & x<0 \\ 1-x & 0\leq x\leq 1 \\ 0 & x>1 \end{cases} \tag{33}$$

The solution to this initial-value problem is a compression wave which initially consists of a fan for $0\leq x\leq1$, for the compression wave the fan closes (or compresses) with time until a shock is formed, and the solution is a continuous wave that gets steeper until t=1, when it becomes a shock. The physical solution for $t<1$ is (Smoller, 1988)

$$u(x,t)=\begin{cases} 1 & x<t \\ \frac{1-x}{1-t} & t\leq x\leq 1 \\ 0 & x>1 \end{cases} \tag{34a}$$

and for $t\geq1$

$$u(x,t)=\begin{cases} 1 & x<t \\ \frac{1-x}{1-t} & t\leq x\leq 1 \\ 0 & x>1 \end{cases} \tag{34b}$$

(b) Consider initial data with one contact discontinuity

$$u(x,0)=\begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases} \tag{35}$$

This initial data leads to interesting results. There are two solutions which satisfy the equation, one is a discontinuous function (Smoller, 1988)

$$u_1(x,t)=\begin{cases} 1 & x>0.5t \\ 0 & x<0.5t \end{cases} \tag{36a}$$

and one is a continuous function, which is a rarefaction wave

$$u_2(x,t)=\begin{cases} 0 & x<0 \\ x/t & 0<x<t \\ 1 & x>t \end{cases} \tag{36b}$$

We obtained the approximate solution to the rarefaction wave solution. From case (a), we can see a feature of a nonlinear equation, in which continuous initial data can have a discontinuous solution, and a continuous solution can follow from discontinuous initial values.

(c) Consider initial value with two contact discontinuity fronts (Wu and Hua, 1988)

$$u(x,0)=\varphi(x)=\begin{cases}1 & x\geq 0.5\\ -1 & x<0.5\end{cases} \tag{37}$$

The solution to this initial value is a rarefaction wave with two fans

$$u(x,t)=\begin{cases}\varphi(x), & x<0.5-t \text{ or } x>0.5+t\\ (x-0.50)/t, & 0.5-t\leq x\leq 0.5+t\end{cases} \tag{38a}$$

Wu and Hua (1988) mention MacCormack's finite difference scheme for the weak solution

$$u(x,t)=\varphi(x),\quad t>0 \tag{38b}$$

which does not approximate the physical solution.

Table I Error in L^2-norm for case (a) using N=100 particles.

error(t)	*t*=0.0	*t*=0.5	*t*=1.0	*t*=1.5	*t*=2.0	*t*=2.5
case (a)	0.00103	0.00202	0.00350	0.01435	0.01435	0.01435

The errors for case (a) are given in Table I and for cases (b) and (c) are given in Table II. Comparison of the convergence speed with change the initial pseudo-compression size using the same number of particles is given in Table III. The errors in L^2-norm for case (a) are shown in Table I. As we can see from Table I, the error for case (a) is like $O(\sqrt{1/N})$ at time $t<t^k$, and the max-error at the shock front is like $O(N^{-4/7})$.

Table II Error in L^2-norm for cases (b) and (c) using N=100 particles.

error(t)	*t*=0.0	*t*=0.5	*t*=1.0	*t*=1.5	*t*=2.0	*t*=2.5
case (b)	0.55921	0.01106	0.00559	0.00373	0.00280	0.00224
case (c)	0.55921	0.01106	0.00559	0.00373	0.00280	0.00224

In Table II, the errors are decreasing with time even if the errors at t=0 are large. The error should be decrease in time because of the equation's nonlinear character as we described in section 2. The speed of convergence decreases like $O(\Delta t/2n)$.

Table III Error in L^2-norm with decreasing size of the pseudo-compression region using N=100 particles.

error(t)	*t*=0.0	*t*=0.5	*t*=1.0	*t*=1.5	*t*=2.0	*t*=2.5
case (b)	0.55921	0.00559	0.00280	0.00186	0.00133	0.00112
case (c)	0.52301	0.00523	0.00262	0.00174	0.00131	0.00105

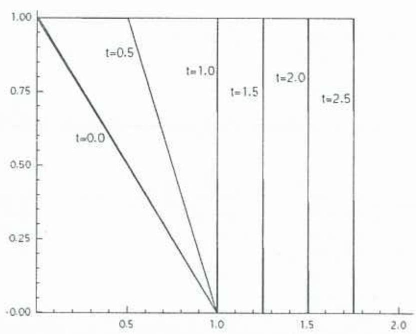

Figure 1 Compression wave solutions for case (a) showing the formation of the shock at t=1.

Table III shows that the convergence speeds for cases (b) and (c) depend on the size of initial pseudo-compression region $[x_0, x_0+1/jN]$, and illustrates that the rate of convergence is like $O(\Delta t/2jn)$. Figures 1, 2 and 3 illustrate the nonlinear behaviour of each case, such as the shock and the expansion phenomena, and compare the approximate and exact solutions.

Conclusion: The solutions of a nonlinear equation with different types of initial values are approximated by a particle method. The accuracy for the compression wave is $O(\sqrt{1/N})$ and for the rarefaction wave is $O(\Delta t/2n)$. Numerous numerical experiments show that this particle mehod used here is accurate, stable, simple, and that the solutions are determined by the initial data, not by "further events".

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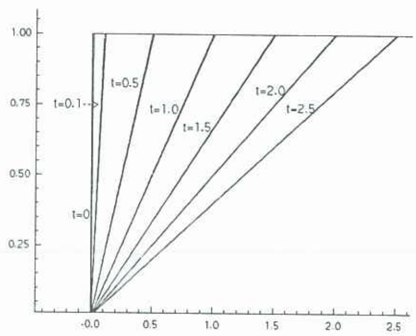

Figure 2 Rarefaction wave solutions for case (b) with 1 fan.

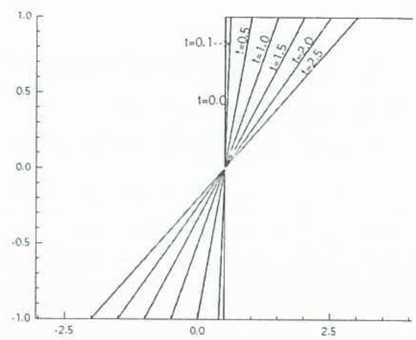

Figure 3 Rarefaction wave solutions for case (c) with 2 fans.