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MODELS FOR THE NONLINEAR DYNAMICS OF LANGMUIR CIRCULATIONS

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ABSTRACT

Langmuir circulations are wind-driven convective motions that arise in the upper layers of oceans and lakes. They are an important mixing mechanism, responsible for an exchange of heat and momentum between atmosphere and ocean. We explore the Craik-Leibovich theory of Langmuir circulations in a density-stratified layer of finite depth, in which the motions arise from an instability of a wind-driven shear flow. With periodic lateral boundary conditions, the problem has translational and reflectional symmetry in the horizontal cross-wind direction (O(2) symmetry). Two slightly different models for the wind-stress on the water surface are considered.

In the first, a constant stress is applied to the ocean surface due to the difference between ocean and air speeds. Here, the theory does not select a width for the spacing of "windrows" (streaks), a surface manifestation of Langmuir circulations. We impose this width consistent with observation. We describe numerical simulations of Langmuir circulations, and compare them with a weakly nonlinear expansion developed near the onset of motion. This expansion predicts, according to the values of the parameters, stable steady states, travelling waves or modulated waves. All are found in the numerical simulations in excellent agreement with the theoretical prediction. A Takens-Bogdanov bifurcation, at which the linear operator has two vanishing eigenvalues, organises the dynamics over a wide range of parameter values. Another multiple bifurcation, the simultaneous overstability of two Langmuir circulation modes with windrow spacings in the ratio 2:1, is analysed, but proves less useful.

The second wind-stress model accounts for the reduction in the applied stress when the ocean surface accelerates in the wind direction, and predicts a finite spacing of the windrows at onset. We discuss the linear and nonlinear selection mechanisms for windrow spacing.

INTRODUCTION

We describe here two approaches to deriving models of Langmuir circulations (Langmuir, 1938), both based on the Craik-Leibovich theory of these convective motions. (See Leibovich (1983) and references therein for details of the phenomenon.) The first approach, a weakly nonlinear theory, generates a small number of ordinary differential equations for the small amplitudes of the motions near onset. In the second, a small partial differential system is derived through a shallow-water approximation.

THE MODEL

According to the Craik-Leibovich theory, Langmuir circulations arise as an instability of a wind-driven shear flow, and result from the interaction between this shear flow and the Stokes drift due to surface gravity waves. The motions are assumed to be confined to the mixed layer, of finite depth, in which the water may be density-stratified. Equations of motion for mean flow quantities have been derived, where the mean is taken over the short timescales of the surface waves. Details may be found in Cox *et al.* (1992a) and references therein.

Observations indicate that Langmuir circulations often have a longitudinal lengthscale which is an order of magnitude greater than their width, and so we restrict our attention to motions that are two-dimensional, that is, independent of the wind-directed co-ordinate, x.

The basic state is a linear velocity profile, $\boldsymbol{U} = (U_0 + U_1 z/d, 0, 0)$, and a linear temperature profile, $T = T_0 + T_1 z/d$. Here, z is the vertical co-ordinate, d is the depth of the mixed layer, and the horizontal cross-wind co-ordinate is y. Langmuir circulations are perturbations to this basic state.

We shall deal in what follows with dimensionless quantities: the details of the rescalings involved are given by Cox *et al.* (1992a). In particular the depth of the mixed layer is normalised to 1, so that $0 > z > -1$. The mean ocean surface is at $z = 0$, and the bottom of the mixed layer at $z = -1$.

Governing Equations For The Pertubations

The cross-wind and vertical velocities may be expressed in terms of a streamfunction ψ, that is, $(v, w) = (\psi_z, -\psi_y)$. We denote the perturbation to the wind-directed shear flow by u, and the perturbation to the linear temperature profile T by θ. The perturbations to the basic state are then governed by the partial differential equations

$$\begin{aligned} (\partial_t - \nabla^2)\zeta &= Rh(z)u_y - S\theta_y + \psi_y\zeta_z - \psi_z\zeta_y, \\ (\partial_t - \nabla^2)u &= \psi_y + \psi_y u_z - \psi_z u_y, \\ (\partial_t - \tau\nabla^2)\theta &= \psi_y + \psi_y\theta_z - \psi_z\theta_y, \end{aligned} \qquad (1)$$

where $\zeta = \nabla^2\psi$ and $\nabla^2 = \partial_y^2 + \partial_z^2$. The function $h(z)$ is the dimensionless Stokes-drift gradient, and we take $h(z) = 1$ for this paper. The parameters are: a destabilising Rayleigh number, $R = (U_1 d^3/\nu_T^2)\,\partial U_s(0)/\partial z$, that indicates the interaction between the shear flow and the Stokes drift, given

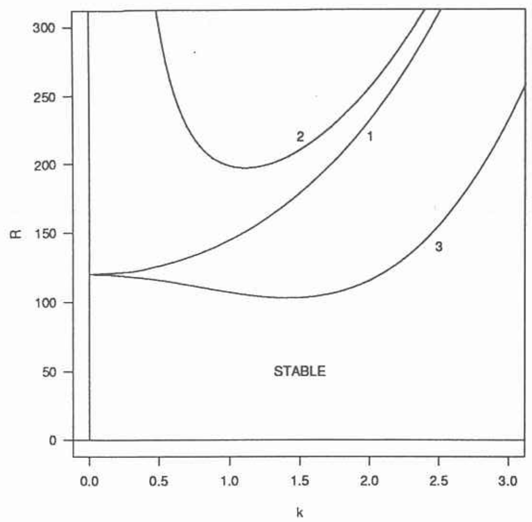

Figure 1: *Linear stability of the basic state* $(\boldsymbol{U}, T)$. $R_c(k)$ *is plotted against wavenumber* k *for three different choices of parameters and boundary conditions. Curve 1 has* $S = 0$ *and simple boundary conditions from case (i). Curve 2 has* $S = 0$ *and typical boundary conditions from case (ii)* $[\alpha_t = 0.06$ *and* $\alpha_b = 0.28]$. *Curve 3 has the boundary conditions (i), together with thermal boundary conditions* $\theta = 0$ *at* $z = 0, -1$; $S = -50 < 0$ *so the layer is* unstably *stratified.*

by $(U_s(z), 0, 0)$; a thermal Rayleigh number, $S = \beta g T_1 d^3/\nu_T^2$, that for the present work we take to be positive, and therefore stabilising (warmer water over cooler); an eddy Prandtl number, $\tau = \alpha_T/\nu_T$. Here β is the coefficient of volume expansion, g the gravitational acceleration, ν_T an eddy viscosity, and α_T an eddy diffusivity of heat. For our numerical results, $\tau = 1/6.7$.

Equations (1) are mathematically analogous to the equations of double diffusion (Cox *et al.*, 1992a).

Linear Stability

The linear stability of the basic state is determined by considering infinitesimal disturbances which are normal modes of the form

$$(\psi, u, \theta) = (\hat{\psi}(z), \hat{u}(z), \hat{\theta}(z)) \exp(iky + \sigma t).$$

The value of R at which the basic state becomes unstable to disturbances of wavenumber k is denoted by $R_c(k)$. The wavenumber for which $R_c(k)$ is minimised is denoted by k_c. This is the wavenumber of the first mode to become unstable as R is increased.

Boundary Conditions

The boundary conditions on $z = 0, -1$ are an important aspect of the model for Langmuir circulations. They significantly affect the linear stability of the basic state (Cox & Leibovich, 1992). Two distinct choices have been examined, one relatively simple, and the other somewhat more sophisticated. Each assumes no flux of fluid through the top and bottom of the mixed layer.

(i) The simpler choice for the remaining boundary conditions assumes a constant applied wind stress at the ocean surface, and constant stress at the bottom of the mixed layer. The perturbations are then subject to boundary conditions of no stress and no flux. In particular the x-velocity perturbation satisifes the boundary conditions

$$\partial u/\partial z = 0 \quad \text{at } z = 0, -1. \tag{2}$$

A consequence of these boundary conditions, in particular of (2), is that the first mode to be destabilised has infinite wavelength, that is, $k_c = 0$. The same behaviour is found in Rayleigh-Bénard convection between non-conducting surfaces (Sparrow *et al.*, 1964). A typical marginal stability curve is shown in Figure 1 (curve 1).

(ii) More sophisticated boundary conditions have been derived that, *inter alia*, account for the mechanism of stress transmission by the wind. For example, at the ocean surface we impose the condition

$$\partial u/\partial z + \alpha_t u = 0, \quad \alpha_t > 0,$$

which recognises that the stress results from the difference between wind and ocean-surface speeds, so that an acceleration of the ocean surface is balanced by a reduction in the applied wind-stress. Similar mixed boundary conditions are applied at the lower boundary $z = -1$, and for ψ. Details are given by Cox & Leibovich (1992). A consequence of this improvement in the modelling of the boundary conditions is that the widest modes are stabilised, and so $k_c > 0$. Typically, α_t and the corresponding constant appropriate at the bottom of the mixed layer, α_b, are small. In the limit as $\alpha_t, \alpha_b \to 0$, $k_c = O(\alpha^{1/4})$, where $\alpha \equiv \alpha_t + \alpha_b$. Typical values of α_t and α_b are estimated by Cox & Leibovich (1992), and a typical marginal stability curve is shown in Figure 1 (curve 2). Note that $k_c > 0$.

The boundary conditions described in case (i) are a special case of the more general boundary conditions (ii), and are given by setting $\alpha_t = \alpha_b = 0$.

The boundary conditions on temperature play a less important role in selecting the width of the Langmuir circulation cells. They are, however, important in determining whether convection is steady or oscillatory ($\sigma = 0$ or $\sigma = i\omega$ at marginal stability, respectively).

If the mixed layer is sufficiently *unstably* stratified then even for $\alpha_t = \alpha_b = 0$ the first mode to be destabilised may have $k_c > 0$. In this case the thermal instability is more important than the mechanically-driven instability. Such an event is illustrated in Figure 1 (curve 3). Similar behaviour in binary-fluid convection has been predicted by Knobloch & Moore (1988).

The lateral boundary conditions are periodic in y with period L. This endows the system with O(2)-symmetry.

WEAKLY NONLINEAR ANALYSIS

Boundary Conditions (i)

Here a width for the circulations must be imposed. Square rolls with $k = \pi$ and $L = 2$ (as wide as the mixed layer is deep) have been investigated by Leibovich, Lele & Moroz (1989), and by Cox *et al.* (1992a,b).

Linear stability analysis indicates that for slight or no density stratification the basic state becomes unstable to steady convection, while for larger values of S oscillatory convection is predicted. One particular value for the parameters, $(R, S) = (R_d, S_d) \approx (741.64, 72.01)$, marks the boundary between the two instabilities, where the linear operator in (1) has a pair of zero eigenvalues. A small-amplitude analysis of the weakly nonlinear dynamics of the system near this "Takens-Bogdanov" bifurcation predicts for $S > S_d$ stable travelling waves, stable modulated

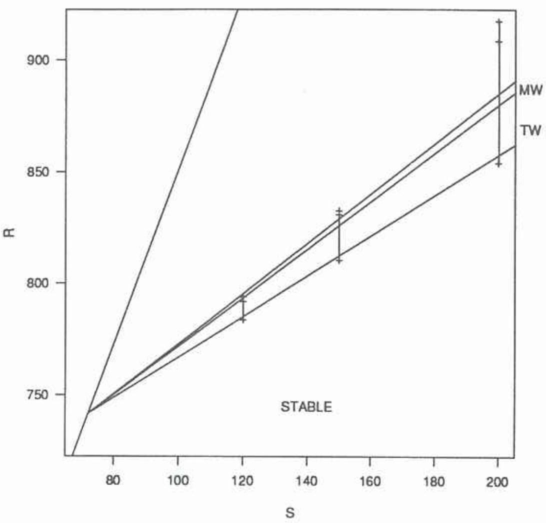

Figure 2: *Stability of the basic state near the Takens-Bogdanov bifurcation.*
Oblique lines represent theoretical predictions: travelling waves (TW) and modulated waves (MW) are predicted to be stable in the wedges of parameter values indicated. Vertical lines at $S = 120, 150, 200$ indicate where full numerical simulations of (1) yield TW (between lower pair of "+" marks) or MW (between upper pair). Agreement between theory and simulations is best, as expected, near the bifurcation point.

travelling waves, and large-amplitude steady states as R is increased (Cox *et al.*, 1992a). Figure 2 indicates the regions in (R, S)-parameter space for which the various solutions are predicted to be stable. Numerical simulations of the full partial differential equations (1) near the bifurcation point agree very well with the theoretical predictions, and are indicated for comparison in Figure 2. In fact, when $L = 2$ all numerical solutions of (1) over a very wide range of parameter values can be traced back to this multiple bifurcation point: the Takens-Bogdanov bifurcation organises the dynamics.

If we impose instead the spatial period $L = 4$, so the fundamental Langmuir circulation cells are twice as wide as deep, the simultaneous Hopf bifurcation of the fundamental ($k = \pi/2$) and its first harmonic ($k = \pi$) takes place when $(R, S) = (R_h, S_h) \approx (1613.3, 1078.9)$. A weakly nonlinear analysis near this multiple bifurcation point predicts that the only stable small-amplitude solution is a travelling wave that occurs in a very thin wedge of parameter values. This double-Hopf bifurcation does not organise the dynamics of the system with $L = 4$ (Cox *et al.*, 1992b). Indeed we were unable to find the predicted travelling wave in our numerical simulations of (1) due to the extremely small region in parameter space in which it is predicted to be stable. (However, if no-flux boundary conditions are applied at $y = 0$ and $y = 4$ then standing waves are stabilised in some regions of parameter space.)

Boundary Conditions (ii)

With boundary conditions (ii), a natural width, π/k_c, for the Langmuir circulation cells is indicated by the linear theory, and so L need not be externally imposed.

Depending on the value of the stratification parameter, S, and the thermal boundary conditions, there may be either steady or oscillatory convection.

Steady convection. Provided the stratification is not too great (or if the layer is unstably stratified) then steady convection takes place for $R \sim 120$, and an evolution equation for large-wavelength disturbances, valid without restriction on the amplitude of the motions, may be derived in the physically interesting limit of small α. In that limit, $k_c = O(\alpha^{1/4})$, and an expansion in the small wavenumber yields a single partial differential equation for $\bar{u} \equiv \int_{-1}^{0} u(y, z, t)dz$:

$$\partial_t \bar{u} = -\alpha\bar{u} - \Delta R \partial_y^2 \bar{u} - a\partial_y^4 \bar{u} + b\partial_y (\partial_y \bar{u})^3, \qquad (3)$$

where $\Delta R = (R - 120)/120$, and the constants a and b depend on the thermal boundary conditions and S/τ. When $S = 0$, $a = 1091/5544$ and $b = 155/126$ (Chapman & Proctor, 1980). If S is sufficiently negative (so the basic state is statically unstable) then a may become negative, and terms of order ∂_y^6 must be included in (3) to stabilise the short-wavelength disturbances.

An equation analogous to (3) has been derived for other convective systems (*e.g.*, Chapman & Proctor, 1980; Sivashinsky, 1982). Equation (3) has a Lyapunov functional $V[\bar{u}]$ (Chapman & Proctor), and therefore the asymptotic behaviour cannot be time-dependent. Minimising $V[\bar{u}]$ offers a nonlinear selection mechanism for the wavelength of convection.

It is easy to see directly that travelling waves are forbidden as solutions of (3), for if $\bar{u}(y,t) = f(\eta)$, where $\eta = y - ct$, then, multiplying (3) by f', integrating over the interval $0 < y < L$, and applying periodic boundary conditions, we find

$$-c\int_0^L (f')^2 \, dy = 0.$$

Therefore either f is trivially zero, or $c = 0$.

We have integrated (3) numerically with periodic boundary conditions on an interval $0 < y < 100$, using a spectral code. We find that there may be multiple stable steady states. For example, with typical values for α_t and α_b (0.06 and 0.28, respectively), no stratification, and $R = 208$ (so $\Delta R \approx 0.73$) we find that the solutions with between 10 and 17 pairs of Langmuir circulation cells are all stable to small disturbances. In this example, the Lyapunov functional is minimised for the solution with 13 pairs of cells, which corresponds to a "preferred" wavenumber of $k \approx 0.8168$. Linear theory gives a critical wavenumber for the onset of convection as $k_c \approx 1.146$, so the nonlinear theory predicts wider Langmuir circulation cells than the linear.

Figure 3 compares the computed Nusselt numbers from (1) and (3) with $\alpha_t = \alpha_b = 0.01$. The Nusselt number is a dimensionless quantity that measures the heat transport across the ocean surface—when there is no convection $Nu = 1$. The linear results are captured very well: from (1) we obtain $k_c = 0.5604$ and $R_c = 135.788$, while from (3) the corresponding values are 0.5646 and 135.057. In this example, the one-dimensional model (3) tends to underestimate the critical Rayleigh number R_c of the full system, and to overestimate the heat transport.

Oscillatory convection. When the thermal boundary conditions are of no flux,

$$\partial\theta/\partial z = 0 \quad \text{at } z = 0, -1,$$

and $S > 120\tau^2/(1 - \tau)$, an oscillatory (Hopf) bifurcation occurs in (1) for $R \sim R_0 = 120 + 120\tau + S$. The long

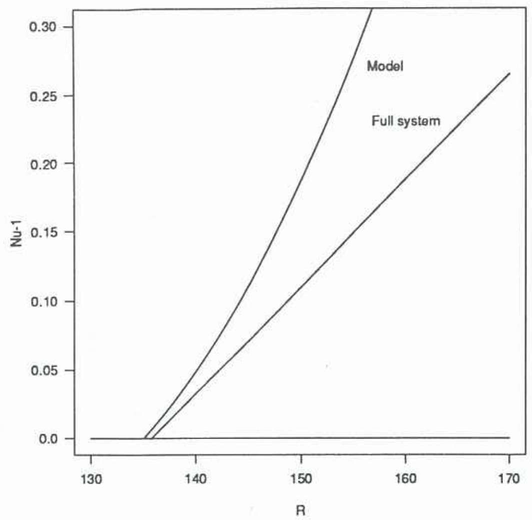

Figure 3: *Plot of $Nu-1$ against R for the unstratified case, $S=0$.*
The upper curve is given by the model (3), while the lower curve represents numerical integrations of (1). [$\alpha_t = \alpha_b = 0.01$.]

wavelength of the motions may be exploited to derive a pair of evolution equations for $\bar{u}$ and $\bar{\theta} \equiv \int_{-1}^{0} \theta(y,z,t)dz$ near this bifurcation:

$$\begin{aligned} \bar{u}_t &= -\tau(1+\Sigma)\partial_y^2\bar{u} + \tau\Sigma\partial_y^2\bar{\theta} - \alpha\bar{u} - \Delta R\partial_y^2\bar{u} \\ &\quad - a\partial_y^4\bar{u} + b\partial_y^4\bar{\theta} + \tau c\partial_y\left\{[\partial_y(R_0\bar{u} - S\bar{\theta})]^2\partial_y\bar{u}\right\} \\ \bar{\theta}_t &= -(1+\tau(1+\Sigma))\partial_y^2\bar{u} + \tau(1+\Sigma)\partial_y^2\bar{\theta} - \Delta R\partial_y^2\bar{u} \\ &\quad - a\partial_y^4\bar{u} + b\partial_y^4\bar{\theta} + c\partial_y\left\{[\partial_y(R_0\bar{u} - S\bar{\theta})]^2\partial_y\bar{\theta}\right\}. \end{aligned}$$

The coefficients are

$$\begin{aligned} a &= \frac{1}{5544}\left\{530\tau^2 + 1621\tau + 1091\right. \\ &\quad \left. + \Sigma(499\tau^2 + 1091\tau + 31) + 31\Sigma^2\tau(1-\tau)\right\} \\ b &= \frac{\Sigma}{5544}\left\{1652\tau - 31\tau^2 + 31\Sigma\tau(1-\tau)\right\} \\ c &= \frac{31}{362880\tau}, \end{aligned}$$

where $\Delta R = (R - R_0)/120$ and $S = 120\tau\Sigma$. These equations admit both travelling and standing waves. We examine here the TW. These take the form $\bar{u} = f(\eta)$, $\bar{\theta} = g(\eta)$, where $\eta = y - pt$. Near their bifurcation from the basic state, the TW are approximately sinusoidal,

$$f(\eta) \sim Fe^{ik_c\eta} + c.c., \quad g(\eta) \sim mFe^{ik_c\eta} + c.c.,$$

where

$$\begin{aligned} k_c^4 &= \alpha/(a-b), \\ m &\sim \left[R_0 - 120 + 120ip/k_c\right]/S, \\ p^2 &\sim \left[S(1-\tau)/120 - \tau^2\right]k_c^2. \end{aligned}$$

A weakly nonlinear expansion of the solution, to be published in detail elsewhere, shows that the amplitude of the TW satisfies

$$|F|^2 = \frac{R - R_c}{120k_c^2cR_0(1-\tau)},$$

where

$$R_c = R_0 + 240\sqrt{\alpha(a-b)}.$$

We see that the bifurcation is supercritical (that is, there are TW for $R > R_c$).

These expressions are valid provided $a - b > 0$, that is, provided either $530\tau - 31 < 0$ or

$$S < 120\tau\frac{(1+\tau)(530\tau + 1091)}{(1-\tau)(530\tau - 31)}.$$

CONCLUSIONS

We have described two approaches to deriving models of nonlinear convection according to the Craik-Leibovich theory for Langmuir circulations. In the first an aspect ratio for the cells must be chosen, while in the second the model itself selects the aspect ratio. The assumption that k is small in the second model proves in practice not to be too restrictive because the results derived under this assumption are quantitatively accurate even for moderate, and realistic, values of the horizontal wavenumber (see Cox and Leibovich (1992) for more details).

We have given partial differential equations for both steady and oscillatory convection under a small-wavelength assumption. Near the steady bifurcation, travelling waves are forbidden as solutions of the nonlinear evolution equation, but they may occur at the oscillatory bifurcation, where they are supercritical. It remains to be determined whether travelling waves or standing waves are preferred.

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