11th Australasian Fluid Mechanics Conference
University of Tasmania, Hobart, Australia
14-18 December 1992

THE STABILITY OF VORTEX DIPOLES

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SUMMARY

Numerical simulations of vortex dipoles have been performed in order to examine which functional relationship $\omega = f(\psi)$ they assume at the steady state. The ω, ψ–relationships of the various dipolar structures are compared with that of the non-perturbed Lamb dipole, which serves here as a reference. We have found that the entropy, defined by Robert and Sommeria (1991), is a global quantity which characterizes the steady state very well. As a more realistic perturbation the case of two dipoles colliding at a certain angle has been considered. With this case it is possible to explain certain features observed in experiments by van Heijst and Flór (1989,1992) and in numerical simulations by Montgomery *et al.* (1992).

INTRODUCTION

Vortex dipoles are common features of geophysical flows, and they play an important role in their dynamics. In the ocean, dipolar vortices may be generated in various ways (e.g. as a result of shedding from unstable boundary currents or due to localized wind forcing) and they provide an important mechanism in the transport of physical properties (heat, pollutants). In the atmosphere, dipolar flow structures in the form of blocking systems tend to have a stabilizing influence on the local weather. Within the context of stability of such flow structures, it is important to know whether the structure, once perturbed, relaxes towards its initial (stable) state. It is easy to show that any functional relationship $\omega = f(\psi)$ between the vorticity ω and the streamfunction ψ satisfies $J(\omega, \psi) = 0$ and is thus a stationary solution of the inviscid Euler equations. In the present study we consider the Lamb dipole, which travels with a constant velocity while preserving its shape. For this dipolar vortex the relation between ω and ψ, in a reference frame travelling with the dipole, is linear, i.e. $\omega = k^2\psi$.

Analytical approaches based on linear approximations are not very helpful to study the stability of dipoles, as it has recently been shown by Nycander (1992) for modons, which are quasi-stationary dipoles on a *beta-plane*. Numerical approaches are more useful, because different types of perturbations can be artificially imposed to see whether the dipole relaxes to a new steady configuration and, if so, which functional relationship $f(\psi)$ is yielded. It is of fundamental importance to know the global quantity which actually characterizes the steady configuration. Usually, numerical simulations are based on schemes that conserve both energy and enstrophy, and the check of the preservation of these quantities for the Euler equations is then meaningless, of course. Recently Sommeria *et al.* (1991) used the *entropy*, defined by Robert and Sommeria (1991) to study the evolution of a mixing layer. In the present work, we have considered a Lamb dipole subjected to different types of perturbations in order to examine which functional relationship is finally reached, and it is found that the entropy defines the achievement of the steady state satisfactorily. Also, it has been found that the changes of the vortical structures at low levels of vorticity, which do not alter enstrophy and energy, produce a modification of the distribution of the entropy.

In the simulations it has been observed that once the steady state is reached the ω, ψ–relationship is piece-wise linear, with different slopes. Near the vorticity peaks, the slope is equal to that of the unperturbed Lamb dipole, while the parts of the scatter plot corresponding with the dipole's edge and axial regions have a slope which is definitely different from that of the Lamb dipole, depending on the distance between the cores of the two vortices. The occurrence of different slopes in the relationship $\omega = f(\psi)$ is possibly connected with the relationships that have been observed experimentally by van Heijst and Flór (1989,1992) and in numerical simulations by Montgomery *et al.* (1992), where a *sinh* relationship was suggested.

EQUATIONS AND RESULTS

We have performed two-dimensional numerical simulations by solving the Navier-Stokes equations in vorticity/streamfunction formulation. The calculations were performed with a finite-difference scheme, as described in Orlandi (1989). The scheme is second-order accurate in space, it uses the Arakawa scheme for the nonlinear terms, preserving energy, enstrophy and the skew symmetry of the Jacobian in the inviscid case. A third-order Runge Kutta scheme, with two-level storage, explicit for the nonlinear terms, has been used for time advancement. The equations have been solved with a 192×192 grid in a domain periodic in the x_1 direction $(-5 < x_1 < 5)$, which is taken parallel to the translation direction of the vortical structure. In the other direction, $x_2(-5 < x_2 < 5)$, a symmetry condition is used, with the boundaries located sufficiently far away from the dipole to ensure that the effects of the image dipoles are negligible. In the inviscid case, perturbed dipoles leave behind a certain amount of the initial vorticity, before they reach the steady state. To avoid the collision of the dipole with the shed vorticity present in its own wake when periodically re-entering the domain from the left boundary, a

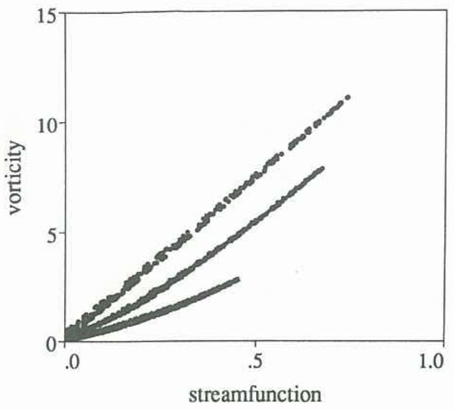


Fig.1 Scatter plots for an unperturbed, viscously-decaying Lamb dipole at $Re = 500$ for $t = 0.$, $t = 12.$, $t = 60.$

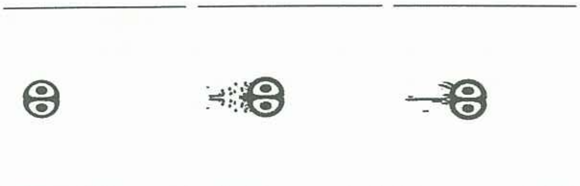


Fig.2 Contour plots of entropy for an unperturbed Lamb dipole at (a) $t = 0.$, (b) $t = 11.$, (c) $t = 96.$

cleaning procedure for the ejected vorticity has been used. With this procedure the small-scale vorticity in the dipole's far wake is eliminated before the dipole leaves the domain or re-enters it. Whenever the cleaning is performed, a new streamfunction related only to the remaining vorticity is evaluated.

Different perturbations on the shape of the dipole have been considered and the evolution has been compared to that of the Lamb dipole, which theoretically is a steady vorticity configuration.

The Lamb dipole has an initial distribution $\omega = k^2\psi$ throughout the recirculation domain: the centres of the two vortices are separated by a distance $l = 1$ and, in the inviscid case, the dipole moves with a constant velocity. Circulation, kinetic energy and enstrophy have been calculated, in order to verify their preservation through the numerical scheme. A further quantity that has been evaluated is the entropy, (which is not conserved by the scheme), as introduced by Robert and Sommeria (1991). These authors related the entropy S to the probability $e(\mathbf{x})$ of finding the vorticity level ω in a small neighbourhood of the location $\mathbf{x}$, which is $e(\mathbf{x}) = \omega/a$, where a is the peak vorticity of the dipole. The probability of finding the vorticity level 0 is the complementary $1 - e(\mathbf{x})$. The entropy is given by:

$$S = -\int (e\, loge + (1-e)log(1-e))\, d\mathbf{x} \qquad (1)$$

Since we are interested in finding the steady state, the calculations have been performed in the inviscid case, because in the presence of viscosity the energy and the enstrophy both decrease in time, while the entropy increases. The proof that no steady configuration is reached in the viscous case is given in Fig.1. This graph presents the scatter plots of vorticity versus streamfunction for the Lamb dipole at $t = 0$, $t = 12$ and $t = 60$, showing that at each time a different single-valued relationship is obtained and that for later times $f(\psi)$ tends very closely to the $sinh$ relationship which was observed in other viscous simulations by Montgomery *et al.* (1992) and in laboratory experiments by van Heijst and Flór (1989,1992). In the inviscid case, contour plots of entropy density $\sigma = eloge + (1-e)log(1-e)$ for the Lamb dipole are represented in Fig.2 at $t = 0$, $t = 11$ and $t = 90$. The entropy density is formed by two contributions, which are initially large within the dipole, in particular near the vorticity peaks and in a small region between the rotational and the irrotational flow. Fig.2 shows that near the vorticity peaks σ does not change in time, while a certain amount of σ is shed, and it is related to the low vorticity levels in the wake. A small amount of circulation is generated during the dipole translation and it does not contribute to the energy and enstrophy levels. From plots of the positive circulation it might seem that the numerical scheme does not conserve circulation; this is not true, because the total circulation actually remains zero. It is interesting to observe that the entropy increases during the translation of the dipole and that, at each cleaning of the wake, the entropy decreases and successively increases with the same amount (Fig.3). This mechanism persists during the whole calculation, thus one can assess that the vortical structure is in a steady state condition, in fact the scatter plot at $t = 96$ (Fig.4) follows the same functional relationship as the one corresponding to $t = 0$. The calculation of the Lamb dipole has been performed to check our numerical method and to understand how the entropy evolves.

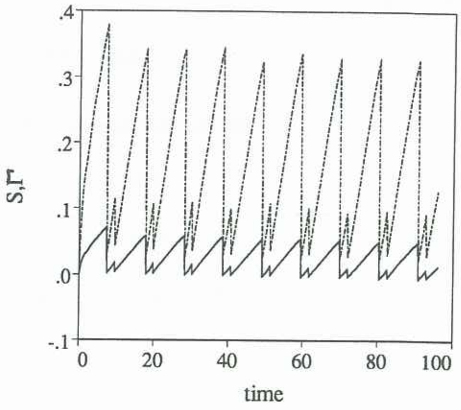


Fig.3 Percentage evolution in time for the unperturbed Lamb dipole shown in Fig.2, ——— circulation: – – – entropy.

When the centres of the two vortices are separated by a distance $l = 2$ the dipole is not any longer in a steady configuration. At $t = 0$ the vorticity is not a single-valued function of ψ and, as a consequence, the dipole redistributes its vorticity to reach the steady configuration (Fig.5). In fact, between $t = 0$ and $t = 11$, when the first cleaning has not been performed, there are substantial changes in the contour plots of the entropy density, which forms a large wake. Following the time evolution of the global quantities, it is observed that the energy and the enstrophy slowly decrease after each cleaning, due to the small amount of vorticity shed from the dipole. A comparison between the entropy evolution in the unperturbed (Fig.2) and in the perturbed cases shows that the perturbation produces a large increase in the first period (Fig.5). In Fig.6 it is seen that, at each cleaning, the global entropy

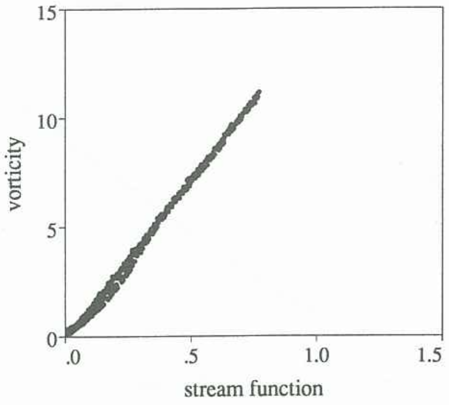


Fig.4 Scatter plot for the unperturbed Lamb dipole shown in Fig.2, at $t = 60$.

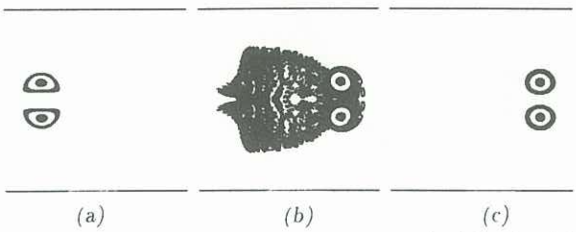


Fig.5 Contour plots of entropy for a perturbed dipole with a separation distance between the vortices $l = 2.0$, at (a) $t = 0.$, (b) $t = 11.$, (c) $t = 200$.

decreases and it tends towards the condition characterizing the steady state. At $t = 200$ (Fig.7) an approximately linear relationship is maintained in the core of the vortices, with the same slope of the Lamb dipole, while some differences still remain in the regions between the two vortices, where the vorticity level is low.

A further simulation has been performed with the centres of the vortices located at a shorter distance, $l = 0.5$. The reduction in size of the dipole produces a reduction in Γ, due to vorticity annihilation and consequently there is a reduction of the energy of the structure. Since the vortices have the same peak vorticity as in the Lamb dipole, but located at a shorter distance, the dipole initially travels with a greater velocity, but when the vorticity is redistributed the translation velocity is immediately reduced. The functional relationship at $t = 0$ has a larger deformation than it had in the previous case. This condition, far from that of equilibrium, leads to a decrease of the initial enstrophy and energy, the reduction being greater than in the previous case: before the first cleaning the entropy grows very rapidly, but it is substantially reduced after only few cleaning cycles. In the case of this smaller dipole, the enstrophy continues to decrease and the entropy does not reach a satisfactory steady condition, as seen in Fig.8. However, the changes in these global quantities are only few % and are not representative of the fact that the scatter plot at $t = 200$ has reached a single-valued relationship which characterizes the steady state (Fig.9). In this case it is observed that the dipole has reached a configuration with a steeper slope in the region near the axis and that the slope near the peak vorticity is equal to that of the Lamb dipole.

A different perturbation has been given by letting two dipoles to collide, as was done in the experiment by van Heijst and Flór (1989), in which it was seen that the dipoles

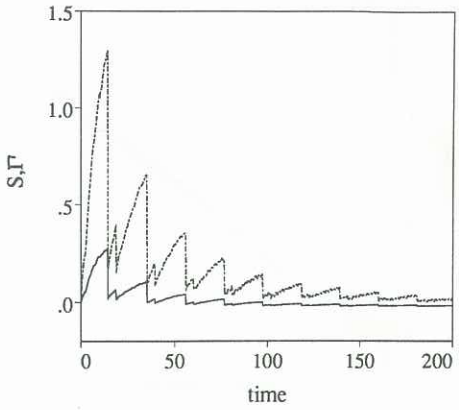


Fig.6 Percentage evolution in time for a perturbed dipole with a separation distance between the vortices $l = 2.0$: —— circulation , − − − entropy.

interchange partners and later on travel again along straight trajectories. In the case of a central collision we observed that the dipole redistributes the vorticity during the impact and that in a short period of time the functional relationship between ω and ψ is the same as before the collision. More interesting is the case of dipoles colliding at an angle. In this case, during the impact the dipoles are largely deformed; the internal dipole looses part of its vorticity, which is sheared by the external dipole, as shown in Fig.10. The numerical simulation has been performed by assuming that the dipole is impacting against a free-slip wall. We considered the collision at an incidence of 45 degrees and we have followed separately the motion of the two vortices, by performing the cleaning as explained before. In the scatter plots in Fig.11 (a)-(b), for the positive and negative vortices, respectively, it is seen that the positive vortex reaches a relationship similar to that obtained when the vortices of the dipoles were separated. Besides, the negative vortex reaches a relationship similar to that of the dipoles with a reduced size: in this case the relationship $\omega = f(\psi)$ shows very clearly the piecewise-linear relationship with two different slopes. We wish to point out that the simulations were performed in the inviscid case: in the viscous case, a smoothing between the two slopes occurs. We have not performed any viscous simulations, because at the moment a comparison of the present numerical simulations with the experimental results can not be realized.

CONCLUSIONS

In the present study several numerical simulations have been performed to understand the stability of vortex dipoles by giving different perturbations to a Lamb dipole. Our results suggest that the region close to the vorticity peak is very stable, while the behaviour in the region close to both the axis and the edge of the dipole changes depending on the perturbation. A fundamental question is: which global quantity characterizes the achievement of the steady state? For the purpose of gaining insight in this matter we have calculated the energy, the enstrophy and the entropy. It was found that energy and enstrophy do not change appreciably, whereas the entropy is very sensitive to small losses of vorticity and thus reaches a steady state with some oscillations. When this stage is reached, the $\omega = f(\psi)$ is a single-valued function. In all the different cases analysed

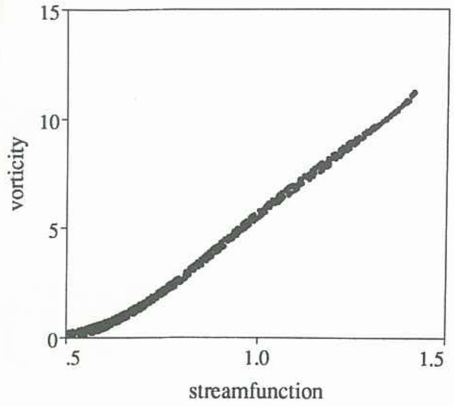


Fig.7 *Scatter plot for a perturbed dipole with a separation distance between the vortices $l = 2.$ at $t = 200$.*

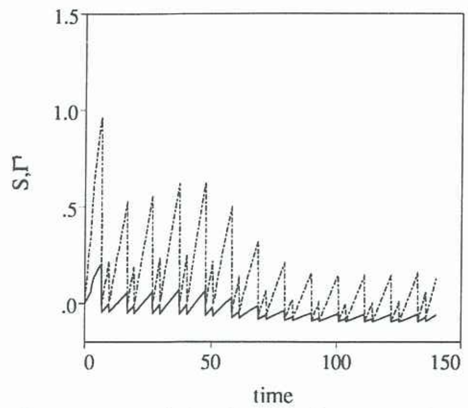


Fig.8 *Percentage evolution in time for a perturbed dipole with a separation distance between the vortices $l = 0.5$: ——— circulation , – – – entropy.*

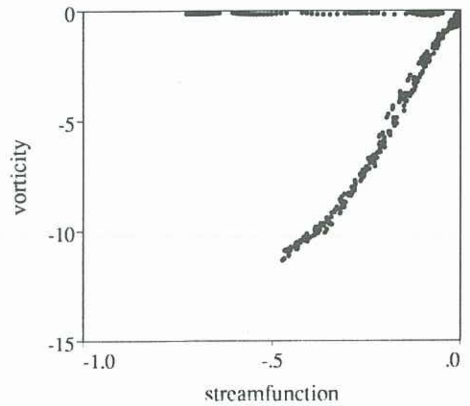


Fig.9 *Scatter plot for a perturbed dipole with a separation distance between the vortices $l = 0.5$ at $t = 120$.*

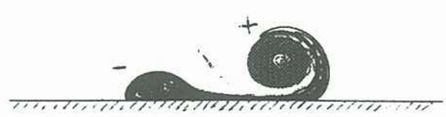
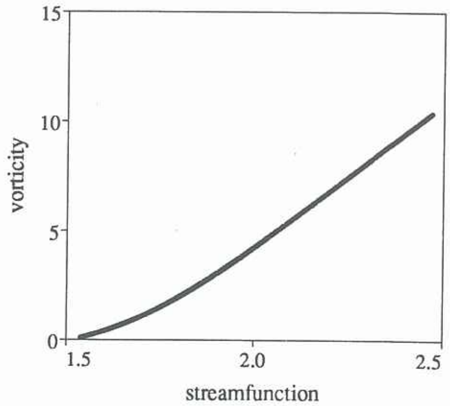
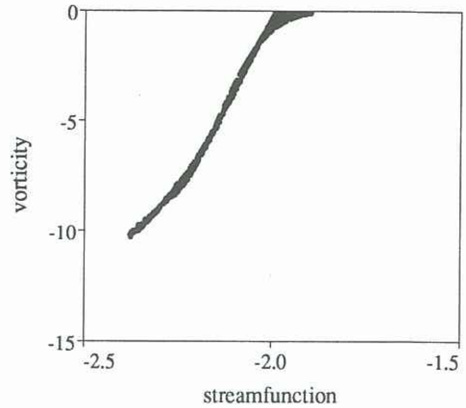


Fig.10 *Contour plot of vorticity for a dipole colliding under 45°against a slip-free solid wall. at $t = 3.7$.*



(a)



(b)

Fig.11 *Scatter plots for the colliding dipole shown in Fig.10, at $t = 50$: (a) positive vortex, (b) negative vortex.*

in these numerical simulations, we observed that the functional relationship is piecewise-linear, with different slopes. Simulations were also performed in the viscous case and it appears that the transition from one slope to the other is well described by a *sinh* relationship, as observed in the experiments by van Heijst and Flór (1989,1992).

REFERENCES

J.B. FLÓR and G.J.F.van HEIJST, (1992) Experiments on dipolar vortices in a stratified fluid, sub. to J.Fluid Mech.

D.MONTGOMERY et.al., (1992) Relaxation in two dimensions and the 'sinh-Poisson' equation, Physics of Fluids, A(4), pp.3-6.

J. NYCANDER, (1992) Refutation of stability proofs for dipole vortices, Physics of Fluids, A(3), pp. 467-476.

P.ORLANDI, (1989) Numerical Simulation of Vortices Motion in Presence of Solid Boundaries, Proc 8th GAMM Conf on Numerical Methods in Fluid Mechanics (1989) pp. 436-441.

R.ROBERT and J.SOMMERIA, (1991) Statistical equilibrium states for two-dimensional flows, J.Fluid Mech., 229, pp. 291-310.

J.SOMMERIA, C.STAQUET and R.ROBERT, (1991) Final equilibrium state of a two-dimensional shear layer, J.Fluid Mech., 233, pp. 661-689.

G.J.F.van HEIJST and J.B.FLÓR, (1989) Dipole formation and collisions in a stratified fluid, Nature, 340, pp. 212-215.