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THE GEOMETRY OF TURBULENT FINE SCALE STRUCTURE

Brian J. CANTWELL

Department of Aeronautics and Astronautics
Stanford University, Stanford CA 94305, USA

ABSTRACT

Recent studies of the topological structure of direct numerical simulations of inhomogeneous turbulence indicate that, in regions of high kinetic energy dissipation rate, the geometry of the local velocity gradient field has a universal character. Namely

(i) Two of the principal rates of strain are positive, one is negative

(ii) The vorticity vector has a tendency to be aligned with the smaller positive principal rate of strain with the probability of perfect alignment approaching one as the data is conditioned on higher and higher rates of dissipation.

(iii) The magnitude of the strain tends to be comparable to the magnitude of the vorticity.

These observations are in agreement with a large number of studies carried out in homogeneous turbulence. In the present paper an attempt to explain this geometry using a restricted Euler model will be discussed. There is some reason to believe that when fully developed equilibrium flows are studied the intermediate principal rate of strain may be negative and this leads to a new, viscous model which is designed to be consistent with the scaling properties of One-Parameter Turbulent Shear Flows.

INTRODUCTION

In the simulations velocity gradients are determined at every point in the flow and used to construct the invariants of the velocity gradient tensor, the rate-of-strain tensor and the rate-of-rotation tensor. The first invariant is zero for incompressible flow and tends to be close to zero for the compressible cases studied and so the basic geometry of the local flow is determined by the second and third invariants. Crossplots are used to produce a concise description of the flow in the space of tensor invariants. These plots reveal significant features which would be difficult or impossible to find using standard visual display techniques. A key aspect of the method is the association which can be made between features of the invariant crossplots and local flow patterns in physical space. The paper by Chong, Perry and Cantwell (1990) is essentially a road map for relating tensor invariants to local flow patterns in compressible and incompressible flows. It should be noted that the method can be applied to the gradient tensor of any smooth vector field which may be of interest including the vorticity field, pressure gradient field and concentration gradient field.

To date we have studied the velocity gradient tensor in six numerically simulated free shear flows (Chen et al. 1990, Sondergaard et al. 1991) including the compressible and incompressible mixing layer, plane wake and homogeneous shear flow. The geometry noted above is consistent with a variety of studies carried out in homogeneous turbulence beginning with Ashurst et al. 1987.

Interest in the fine scale structure of turbulence arises from the key role played by viscous dissipation in the transport of kinetic energy. If we decompose an incompressible turbulent flow flow into a mean and fluctuating part

$$u_i = \overline{u}_i' + u_i' \quad ; \quad p = \overline{p} + p' \qquad (1)$$

and substitute into the momentum equation and take the average, the result is

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_k \frac{\partial \overline{u}_i}{\partial x_k} + \frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{u'_i u'_k}}{\partial x_k} - \nu \frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_k} = 0 \qquad (2)$$

In free shear flows away from walls the Reynolds stresses are much larger than the viscous stresses associated with the mean velocity gradient.

$$\frac{\partial \overline{u'_i u'_k}}{\partial x_k} >> \nu \frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_k} \qquad (3)$$

and in modeling, the viscous term is usually dropped. However if we form the transport equation for the turbulent kinetic energy the result is an equation of the form

$$\frac{\partial}{\partial t}\left(\frac{\overline{u_i' u_i'}}{2}\right) + \frac{\partial C_k}{\partial x_k} + \Pi - \varepsilon = 0 \qquad (4)$$

where C_k is a flux vector involving various products of velocity and pressure fluctuations. In free shear flows away from walls it is observed that the dissipation of turbulent kinetic energy scales with production.

$$\Pi = -\overline{u'_i u'_k} \frac{\partial \overline{u}_i}{\partial x_k} \approx 2\nu\, \overline{S'_{ik} S'_{ki}} = \varepsilon \qquad (5)$$

where

$$S'_{ik} = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_k} + \frac{\partial u_k'}{\partial x_i}\right) \qquad (6)$$

denotes a fluctuation in the rate-of-strain tensor. Note that it is only assumed that dissipation and production scale with each other; exact equality only occurs only in the case of homogeneous shear flow.

CLASSICAL SCALING LAWS

The conclusion from the above discussion is that while viscosity plays relatively little role in the transport of mean momentum, it plays a very important role in the transport of kinetic energy. Moreover fluctuations in the rate-of-strain are large compared to the mean. We can see this as follows. Let U_0 and δ be integral velocity and length scales based on the overall flow. In free turbulence velocity fluctuations tend to be some fraction of U_0 independent of the viscosity (cf. equation 3) and so

$$\Pi \approx U_0^3 / \delta \qquad (7)$$

and from (5)

$$\sqrt{\overline{S'_{ik} S'_{ki}}} \approx \left(\frac{U_0}{\delta}\right)\left(R_\delta\right)^{1/2} \quad ; \quad R_\delta = \frac{U_0 \delta}{\nu} \qquad (8)$$

The implication of this result is that dissipation of kinetic energy takes place in regions with characteristic length scale much smaller than δ and that the associated velocity gradients in these regions are very large compared to the mean. This leads to the concept of turbulent microscales and the classical scaling relationships are as follows:

$$\frac{\delta}{\lambda} \approx (R_\delta)^{1/2} \quad ; \quad \frac{\delta}{\eta} \approx (R_\delta)^{3/4} \quad ; \quad \frac{U_0}{v} \approx (R_\delta)^{1/4} \qquad (9)$$

where λ is a Taylor microscale, η is a Kolmogorov length scale, v is a Kolmogorov velocity scale and the assumption is made that $\frac{v\eta}{\nu} = 1$. Note that the gradients associated with either the Taylor or the Kolmogorov microscale are of the same order. This is to be expected since these scales are essentially defined by (8).

$$\left(\frac{U_0}{\lambda}\right) \approx \left(\frac{v}{\eta}\right) \approx \left(\frac{U_0}{\delta}\right)(R_\delta)^{1/2} \qquad (10)$$

and so, at least as far as dimensional analysis is concerned, motions at both length scales make a comparable contribution to the dissipation of kinetic energy. The fact that instantaneous velocity gradient fluctuations are so much larger than the mean is the justification for carrying over to the description of the instantaneous dissipation the above scaling arguements which are based on quantities defined only in terms of an average.

ONE-PARAMETER TURBULENT SHEAR FLOWS

Virtually all simple turbulent free shear flows have two features in common. They are of a simple unbounded geometry and, away from the region where the flow is created, the evolution of length and velocitiy scales is governed by a single integral invariant of the motion. The primary effect of viscosity is to set the size of microscale motions which dissipate kinetic energy as discussed above. Except through possible initial conditions effects viscosity has little or no direct effect on the integral scales U_0 and δ. With viscosity ignored, the overall flow depends only on the invariant. Simple configurations of this type will be referred to as One-Parameter Turbulent Shear Flows. For all such flows including jets, wakes, mixing layers, vortex rings, vortex pairs, plumes etc the evolution of integral scales in time is given by

$$\delta \approx M^{1/m} t^{k} \quad ; \quad U_0 \approx M^{1/m} t^{k-1} \qquad (11)$$

where M is the invariant of the motion with units $[M]=L^m T^{-n}$ and k = n/m. Equation (11) describes the variation of U_0 and δ in time as seen by a Lagrangian observer who convects with some geometrical feature of the large scale motion. In general $\frac{1}{4} \le k \le 1$. See Cantwell (1981) for further discussion and enumeration of flow cases. For our purposes the significant result using (10) and (11) is the following estimate of the behavior of large scale and microscale velocity gradients as seen by a Lagrangian observer.

$$\frac{U_0}{\delta} \approx \frac{1}{t} \quad ; \quad \left(\frac{U_0}{\lambda}\right) \approx \left(\frac{v}{\eta}\right) \approx \left(\frac{M^{1/m}}{\nu^{1/2}}\right) t^{k-3/2} \qquad (12)$$

In free turbulence the velocity gradients associated with the large scale and microscale motions always decrease with time when referred to a Lagrangian observer.

RESTRICTED EULER MODEL

The velocity gradient tensor satisfies a nonlinear evolution equation of the form

$$\frac{dA_{ij}}{dt} + A_{ik}A_{kj} - (A_{mn}A_{nm})\frac{\delta_{ij}}{3} = H_{ij} \qquad (13)$$

where $A_{ij} = \partial u_i / \partial x_j$ and the tensor H_{ij} contains terms involving the action of cross derivatives of the pressure field and viscous diffusion of the velocity gradient.

$$H_{ij} = -\left(\frac{\partial^2 p}{\partial x_i \partial x_j} - \frac{\partial^2 p}{\partial x_k \partial x_k}\frac{\delta_{ij}}{3}\right) + \nu \frac{\partial^2 A_{ij}}{\partial x_k \partial x_k} \qquad (14)$$

The homogeneous case ($H_{ij} = 0$) can be solved in closed form for A_{ij} and a number of the geometrical features of fine scale motions observed in direct numerical simulations are reproduced by the solution (Cantwell 1992). The asymptotic form of the solution is

$$A_{ij} = K_{ij}(r(t))^{1/3} \qquad (15)$$

where r(t) is a known function which becomes infinite in a finite time and K_{ij} satisfies the algebraic equation

$$+K_{ij} + 2^{1/3}K_{ik}K_{kj} - 2^{2/3}\delta_{ij} = 0 \qquad (16)$$

The matrix K_{ij} can be decomposed into a symmetric and an antisymmetric part $K_{ij} = S_{ij} + W_{ij}$ and upon examination it is found that solutions of (16) have the following properties

(ia) Two of the principal rates of strain are positive, one is negative

(iia) The vorticity vector is aligned exactly with the smaller positive principal rate of strain

(iiia) The second invariants of the strain and rotation tensor satisfy

$$Q = -\frac{1}{2}A_{ik}A_{ki} = Q_S + Q_W = -\frac{1}{2}S_{ik}S_{ki} - \frac{1}{2}W_{ik}W_{ki} = -\frac{3}{2^{2/3}} \qquad (17)$$

For initial conditions which lead to large enstrophy $S_{ik}S_{ki} \approx W_{ik}W_{ik}$. It is rather remarkable that such a simple model for the evolution of the velocity gradient tensor, one which takes out the possibility for adjacent fluid elements to affect one another through the pressure and viscous stress field, would have such a close correspondence to direct numerical simulations of the full Navier-Stokes Equations. Furthermore, in the model, diffusion is ignored and velocity gradients increase with time eventually becoming singular. Whereas our previous discussion of One-Parameter Turbulent Shear Flows indicates that microscale velocity gradients should decrease with time when viscosity is taken into account. The question which comes immediately to mind is: How well developed are the simulations and what should we expect for the geometry of the velocity gradient tensor in fully developed flow, ie, flow for which the proportionality (5) and the scaling relationships (11) hold ?

At the present time most investigators doing direct numerical simulations tend to emphasize pushing the Reynolds number up as high as possible. This increases the CPU time required per time step and therefore places a practical limit on the overall time of the simulation. An important consequence of this is that most simulations are terminated while the flow is still in a strongly unstable state far from equilibrium. This is true of virtually all of the simulations of free shear flows which we have studied to date. The problem is evidenced by the fact that the largest gradients in the flow continue to increase in time right up to the end of the simulation.

A MODEL FOR ONE-PARAMETER FLOWS

It extremely difficult to predict how the velocity gradient tensor should evolve in a general flow and this is due to the fact that it is virtually impossible to make any general statements about the terms which appear in H_{ij} given in (14). This is particularly true of the pressure field which is the solution of a Poisson equation and thus depends on the entire flow. We should expect that viscous diffusion would limit the growth of A_{ij} but other than this very little can be said without considering specific cases and even then the problem of identifying general, local properties of the pressure remains a formidible obstacle. The whole question of whether singularities can develop in rotational solutions of the Euler and/or Navier-Stokes equations is an old and still very controversial issue.

To try to get some insight into the behavior of H_{ij} and to see what we might expect for the local velocity gradient tensor geometry in fully developed flows we shall turn to the class of One-Parameter Flows discussed above and consider the corresponding set of similarity solutions (Cantwell 1986) of the form

$$\xi_i = \frac{x_i}{M^{1/m} t^k} \; ; \; U_i(\xi) = \frac{u_i(\mathbf{x},t)}{M^{1/m} t^{k-1}} \; ; \qquad (18)$$
$$P(\xi) = \frac{p(\mathbf{x},t)/\rho}{M^{2/m} t^{2k-2}} \; ; \; \Lambda_{ij}(\xi) = t \; A_{ij}(\mathbf{x},t)$$

The time derivative of the velocity gradient tensor is

$$\frac{dA_{ij}}{dt} = \frac{1}{t^2}\left(-\Lambda_{ij} + (U_n - k\xi_n)\frac{\partial \Lambda_{ij}}{\partial \xi_n}\right) \qquad (19)$$

and the corresponding transport equation in similarity variables is

$$-\Lambda_{ij} + (U_n - k\xi_n)\frac{\partial \Lambda_{ij}}{\partial \xi_n} + \Lambda_{ik}\Lambda_{kj} - (\Lambda_{mn}\Lambda_{nm})\frac{\delta_{ij}}{3} = \Pi_{ij} \qquad (20)$$

where

$$\Pi_{ij} = -\left(\frac{\partial^2 P}{\partial \xi_i \partial \xi_j} - \frac{\partial^2 P}{\partial \xi_k \partial \xi_k}\frac{\delta_{ij}}{3}\right) + \frac{1}{R_\delta}\frac{\partial^2 \Lambda_{ij}}{\partial \xi_k \partial \xi_k} \qquad (21)$$

and $R_\delta = U_0\delta / \nu = M^{2/m} t^{2k-1} / \nu$ is the Reynolds number based on integral length and velocity scales. We want to know how the velocity gradient tensor evolves when we follow a fluid element in space. In practice this would be done by first computing the trajectory of the fluid element in question. Let this trajectory be given parametrically by $\mathbf{f} = (f_1(t)\; f_2(t),\; f_3(t))$. The transformation of flow variables to noninertial coordinates is

$$x'_i = x_i + f_i \; ; \; t' = t \; ; \; u'_i = u_i + \dot{f}_i \; ; \qquad (22)$$
$$p' = p - x_n \ddot{f}_n \; ; \; A'_{ij} = A_{ij} \; ; \; H'_{ij} = H_{ij}$$

and in terms of similarity coordinates

$$\xi'_i = \xi_i + \frac{f_i}{M^{1/m} t^k} \; ; \; U'_i = U_i + \frac{\dot{f}_i}{M^{1/m} t^{k-1}}; \qquad (23)$$
$$P' = P - \xi_n \frac{\ddot{f}_n}{M^{1/m} t^{k-2}} \; ; \; \Lambda'_{ij} = \Lambda_{ij} \; ; \; \Pi'_{ij} = \Pi_{ij}$$

The coefficient of the convective term in (20) transforms as

$$U'_i - k\xi'_i = U_i - k\xi_i + \frac{t\dot{f}_i - kf_i}{M^{1/m} t^{k-1}} \qquad (24)$$

If we take the origin of the primed coordinate system to be attached to the fluid element of interest and at the origin then $U'_i = 0$, $\xi'_i = 0$ and (19) can be written

$$\frac{dA'_{ij}}{dt'} = \frac{1}{t^2}\left(-\Lambda'_{ij} + \left[\frac{kf_i - t\dot{f}_i}{M^{1/m} t^k}\right]\frac{\partial \Lambda'_{ij}}{\partial \xi'_n}\right) \qquad (25)$$

We now make the assumption that the bracketed term in (25) decreases with time and that at large time

$$\frac{dA'_{ij}}{dt'} \cong \frac{1}{t^2}\left(-\Lambda'_{ij}\right) \qquad (26)$$

This is equivalent to an assumption that the particle tends to be entrained into the critical points of the large scale motion where $U'_i - k\xi'_i = U_i - k\xi_i = 0$. For flows with k=1/2 (ie. the round jet) this is exactly true and no assumption is required. For flows with $k \neq 1/2$ the arguement is essentially heuristic. With this assumption equation (20) becomes.

$$-\Lambda_{ij} + \Lambda_{ik}\Lambda_{kj} - (\Lambda_{mn}\Lambda_{nm})\frac{\delta_{ij}}{3} = \Pi_{ij} \qquad (27)$$

where the primes have been dropped in recognition of the fact that all terms in (27) are invariant under changes of the frame of reference. This should be compared with Equation (16) where the most significant difference is a change in sign of the first term which arises from the fact that in this model the velocity gradients are assumed to *decrease* with time consistent with our expectation of fully developed flow. If, in the same spirit that we earlier assumed H_{ij}=0, we now assume Π_{ij}=0 in (27) one finds that Λ_{ij} has the following characteristics

(ib) Two of the principal rates of strain are *negative*, one is positive

(iib) The vorticity vector is aligned exactly with the smaller *negative* principal rate of strain

(iiib) With Π_{ij}=0 we can use (27) to form equations for the double and triple products of Λ_{ij}. Taking traces leads to

$$Q_\Lambda - \frac{3}{2}R_\Lambda = 0 \; ; \; \frac{2}{9}Q_\Lambda^2 + R_\Lambda = 0 \qquad (28)$$

where $Q_\Lambda = -\frac{1}{2}\Lambda_{ik}\Lambda_{ki}$ and $R_\Lambda = -\frac{1}{3}\Lambda_{ik}\Lambda_{kj}\Lambda_{ji}$ are the second and third invariants of Λ_{ij}. Solving (28) $(Q_\Lambda, R_\Lambda) = (0,0)$ or $(Q_\Lambda, R_\Lambda) = (-3,-2)$.Both roots lie on the boundary

$$\frac{27}{4}R_\Lambda^2 + Q_\Lambda^3 = 0 \quad \text{or} \quad Q_\Lambda = -\left(\frac{27}{4}R_\Lambda^2\right)^{1/3} \qquad (29)$$

which separates real and complex solutions (Chong et al 1990) Thus the second invariants of the strain and rotation tensor satisfy

$$Q_\Lambda = -\frac{1}{2}\Lambda_{ik}\Lambda_{ki} = Q_{\Lambda_S} + Q_{\Lambda_W} = \qquad (30)$$
$$-\frac{1}{2}S_{\Lambda_{ik}}S_{\Lambda_{ki}} - \frac{1}{2}W_{\Lambda_{ik}}W_{\Lambda_{ki}} = (0,-3)$$

As in the restricted Euler model, for initial conditions which lead to large enstrophy $S_{ik}S_{ki} \approx W_{ik}W_{ik}$.

This last result is consistent with the Poisson equation for the mean pressure $\nabla^2\overline{p} = \overline{S'_{ik}S'_{ki}} - \overline{W'_{ik}W'_{ik}}$. In the flows considered the mean pressure tends to be fairly uniform and one should expect that $\overline{S'_{ik}S'_{ki}} \cong \overline{W'_{ik}W'_{ik}}$. The result (ib) and its comparison with (i) and (ia) is consistent with the analysis of Betchov (1956) who showed that in isotropic turbulence $\frac{\partial \omega_i \omega_i}{\partial t} \cong -\langle abc \rangle$ where $\omega_i\omega_i$ is the enstrophy and $\langle abc \rangle$ is the mean of the product of principal rates of strain a, b and c. The tendency for the intermediate positive strain to be positive or negative depends on whether the enstrophy is increasing or decreasing with time.

The model for One-Parameter flows has led to an algebraic equation (27) relating Λ_{ij} and Π_{ij}. This provides an opportunity to learn something fairly general about Π_{ij} albeit within the confines of the assumed time dependence of A_{ij} given by (26). The procedure involves first squaring and then cubing both sides of (27). The Cayley-Hamilton theorem is used to reduce higher order products of Λ_{ij} and then the trace of Π^2 and Π^3 is formed. The result is

$$Q_\Pi = Q_\Lambda - 3R_\Lambda - \frac{1}{3}Q_\Lambda^2 \qquad (31)$$

and

$$R_\Pi = -R_\Lambda + Q_\Lambda R_\Lambda - \frac{2}{3}Q_\Lambda^2 - \frac{2}{27}Q_\Lambda^3 - R_\Lambda^2 \qquad (32)$$

Now a rather amazing thing happens. When we square (32) and cube (31), add the two together and factor the result we find

$$\frac{27}{4}R_\Pi^2 + Q_\Pi^3 = \left(\frac{27}{4}R_\Lambda^2 + Q_\Lambda^3\right)\left(1 + Q_\Lambda - R_\Lambda\right)^2 \quad (33)$$

With reference to (29) and noting that the second factor on the right hand side of (33) is squared we conclude that the eigenvalues of Λ_{ij} and Π_{ij} have the same character, ie, if the eigenvalues of Λ_{ij} are complex then the eigenvalues of Π_{ij} must be complex. Similarly if the eigenvalues of Λ_{ij} are real then the eigenvalues of Π_{ij} must be real. Examining Equation (14) we see that, in the model, complex eigenvalues of of Λ_{ij} cannot occur unless the viscous diffusion of Λ_{ij} is included in Π_{ij}.

In the solution of the restricted Euler model discussed earlier the velocity gradient tensor becomes singular in a finite time and the invariants asymptote to $Q = -\left(27R^2/4\right)^{1/3}$; $R > 0$; there is no localized region of attraction in invariant space. In contrast the One-Parameter Flow model just described does seem to define an attractor in Q_Λ, R_Λ space. This is shown in Figure 1 which depicts contours of the right hand side of (33). The zeros of (33) are also indicated on the Figure. While in principle Q_Λ and R_Λ can range over the whole space extremely large values of $(27/4)R_\Pi^2 + Q_\Pi^3$ are required to move much away from from the very steep-sided well depicted in Figure 1. For a fluid particle in trapped in this well the strain-rates and vorticity tend to behave as indicated in items (ib) and (iib).

The attractor (33) is strongly aligned with the trajectory 1+Q-R=0 and it is of interest to note that this relationship between the invariants comes up in the context of two-dimensional flows undergoing uniform out-of-plane straining. The velocity gradient tensor for such a flow is of the form (Jimenez 1992)

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & b \end{bmatrix} \quad (34)$$

where $b = \pm 1$ is the normalized rate of strain in the z-direction. A positive sign indicates out-of-plane stretching (cf. the steady Burgers' vortex) and a negative sign indicates out-of-plane compression for which no steady state exists. It can be easily shown that the invariants of (34) satisfy

$$1 + Q + \mathrm{sgn}(b)\ R = 0 \quad (35)$$

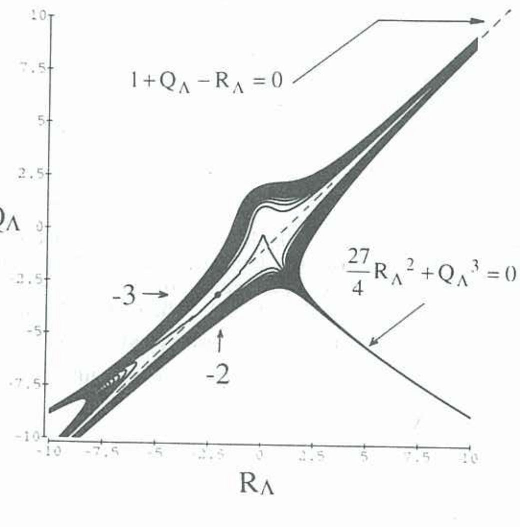

Figure 1

CONCLUDING REMARKS

A new model for the evolution of the velocity gradient tensor has been developed which is designed to be consistent with the time evolution following a fluid particle suggested by the scaling characteristics of One-Parameter Shear Flows. The model contains a fairly localized region of attraction in the space of tensor invariants. The time scale is chosen to be consistent with that of the integral scale motion and a direct connection to the fine scale structure has yet to be made for flows with $k \neq 1/2$. Nevertheless the results indicate that the geometry of the fine scale structure can depend on whether the local velocity gradient tensor is increasing or decreasing with time, an issue which is closely related to the nature of the forces and boundary conditions which create the flow. The results also suggest the need to carry out direct numerical simulations of free shear flows for long times sufficient to reach a fully developed state.

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