11th Australasian Fluid Mechanics Conference
University of Tasmania, Hobart, Australia
14-18 December 1992

RENORMALIZATION GROUP ANALYSIS OF NONLINEAR REYNOLDS STRESS MODELS INCLUDING EFFECTS OF RAPID STRAINING

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ABSTRACT

The Yakhot-Orszag renormalization group is used to develop a nonlinear algebraic Reynolds stress model. Quadratic nonlinearity of the mean velocity gradients produces anisotropy of the normal stresses and admits turbulence driven secondary flows, such as appear in non-circular ducts. When applied to plane channel and square duct flows, good agreement is observed with normal stress measurements in the central 80% of the flow; however, close to the wall the behavior is incorrect, at times even exhibiting negative normal stresses. The non-physical behavior is attributable to the rapid straining in the near-wall region, an effect not included in the model. Further analysis demonstrates the efficacy of allowing the model constants to vary as a function of the local mean strain, thus incorporating a near-wall and rapid distortion effect. The new model is exhibited, the functional variation derived from plane channel flow, and the generalization for three-dimensional flows is discussed.

INTRODUCTION

Laminar flow of a Newtonian fluid in a straight square duct is unidirectional. In contrast, turbulent flow of the same fluid in an identical duct reveals circulation in a cross-section normal to the primary flow direction. Rivlin (1957) suggested an analogy between the laminar flow of a non-Newtonian fluid and the turbulent flow of a Newtonian fluid. Non-Newtonian flow in a non-circular duct produces secondary flows in the cross plane, which Rivlin termed a 'normal stress effect'. The correspondence suggested that a nonlinear generalization of the familiar Boussinesq eddy viscosity might explain the turbulence driven secondary flows. Several investigators have followed this idea of treating turbulence as a viscoelastic medium, for example, Crow (1968), Lumley (1970), and Pope (1975). Speziale (1987) proposed a specific model of this type by requiring that the Reynolds stresses depend on the first two Rivlin-Ericksen tensors.

The first concrete proposal arising from a systematic application of an analytical theory of turbulence was by Yoshizawa (1984). His formula expresses the stresses as an explicit quadratic function of the mean velocity gradients. He uses a two-scale variant of the direct interaction approximation (DIA) of Kraichnan (1959). Yoshizawa's two-scale DIA derivation develops the coefficients of the gradients as integrals of certain combinations of the isotropic Green's function and correlation function of Kraichnan's DIA. A comparison therefore arises with the derivation of transport coefficients from the Boltzmann equation, where Kolmogorov's universal state of locally isotropic turbulence replaces thermal equilibrium as a reference state.

The derivation of an explicit quadratic model by renormalization group (RG) methods leads to a double expansion in tensor product powers of ∇U and ∇U^T, in which the scalar amplitude (coefficient) multiplying each term is a series in powers of ϵ, the expansion parameter of the Yakhot-Orszag theory. Following Yakhot and Orszag (1986), these expansions are truncated at lowest order in ϵ and evaluated at $\epsilon = 0$. Such an explicit quadratic model was derived by Rubinstein and Barton (1990). The same procedure was employed for analysis of the passive scalar and the Reynolds stress transport equations by Rubinstein and Barton (1991, 1992). Whichever formalism is applied, this expansion can certainly be continued beyond the second order and will generate two types of higher order corrections: new nonlinearities, and corrections to existing terms which depend on scalar invariants of the mean velocity field.

In principle, these corrections could be explicitly evaluated perturbatively by TSDIA or RG methods. Although the resulting polynomial model might sometimes be useful, in regions of large $|\nabla U|$ it would be dominated by its highest order terms and could produce inaccurate or non-physical results. Even the explicit quadratic model already predicts negative normal stresses in near-wall flows, as shown by Barton et al. (1991). Yakhot et al. (1992) argue that in such cases, the perturbation expansion in powers of ∇U must be summed, even if only approximately. An analogy to Pade approximation suggests that summation will produce scalar amplitudes which are rational in ∇U; unfortunately, the absence of a simple law of formation for the terms of perturbation theory makes an analytical summation only a remote possibility.

Taulbee (1991) and Speziale (1992) have reached similar conclusions starting from the implicit algebraic Reynolds stress model of Demuren and Rodi (1984), which expresses τ linearly in $\tau\nabla U$ and $\tau\nabla U^T$. Such models can be solved explicitly for τ as a function of ∇U. The result is a model with ten algebraically independent tensor terms in ∇U with coefficients rational in scalar invariants of ∇U. Since the implicit model is finite in the limit of large $| \nabla U |$, the resulting explicit model is also finite in this limit, unlike the explicit quadratic model or its higher order generalizations.

ANALYSIS

We will begin with a model of the type developed by Rubinstein and Barton (1990) and following Yakhot et al. (1992)

$$\overline{u_i u_j} = \frac{2}{3}K\delta_{ij} - C_\nu(\eta)\frac{K^2}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

$$-C_{\tau 1}(\eta)\frac{K^3}{\varepsilon^2}\left[\frac{\partial U_i}{\partial x_p}\frac{\partial U_j}{\partial x_p} - \frac{1}{3}\frac{\partial U_q}{\partial x_p}\frac{\partial U_q}{\partial x_p}\delta_{ij}\right]$$

$$-C_{\tau 2}(\eta)\frac{K^3}{\varepsilon^2}\left[\frac{\partial U_i}{\partial x_p}\frac{\partial U_p}{\partial x_j} + \frac{\partial U_j}{\partial x_p}\frac{\partial U_p}{\partial x_i} - \frac{2}{3}\frac{\partial U_p}{\partial x_q}\frac{\partial U_q}{\partial x_p}\delta_{ij}\right]$$

$$-C_{\tau 3}(\eta)\frac{K^3}{\varepsilon^2}\left[\frac{\partial U_p}{\partial x_i}\frac{\partial U_p}{\partial x_j} - \frac{1}{3}\frac{\partial U_p}{\partial x_q}\frac{\partial U_p}{\partial x_q}\delta_{ij}\right] \quad (1)$$

in which the model coefficients are functions of the simplest, and most experimentally accessible scalar invariant,

$$\eta = \frac{K}{\varepsilon}\left[\left(\frac{\partial U_p}{\partial x_q} + \frac{\partial U_q}{\partial x_p}\right)\frac{\partial U_p}{\partial x_q}\right]^{1/2} = SK/\varepsilon$$

These functions will be chosen so that the model predictions are consistent with known behavior for large η. An interesting theoretical possibility is to require that the model approach the one component limit of rapid distortion theory (RDT) for $\eta \to \infty$. This requires

$$\eta^2 C_{\tau 1}(\eta) \to 2, \eta^2 C_{\tau 3}(\eta) \to 0, \eta C_\nu(\eta) \to 0, \eta \to \infty \quad (2)$$

Such a model could not arise by summing the perturbation theory described above because this theory invokes the isotropic Green's function, whereas the appropriate Green's function for such highly strained flows is itself strain-dependent. Nevertheless, at the level of modeling, introducing a strain-dependent Green's function would only require different η corrections. While this application of RDT is attractive because it fixes the limiting behavior of the model, it should be noted that numerical evidence for the natural occurence of this limit is not conclusive.

A different approach is to find the η dependence from data. The model, eq. (1), can be inverted to yield $C_{\tau 1}, C_{\tau 3}$ in terms of known normal stresses, for simple shear flows. (A simple shear flow is one in which a single mean velocity gradient exists.) For plane channel flow, with $S = \partial U/\partial y$ the only non-vanishing velocity derivative, we obtain

$$C_{\tau 1} = \frac{2b_{11} + b_{22}}{\eta^2} \qquad C_{\tau 3} = \frac{2b_{22} + b_{11}}{\eta^2}$$

where b_{11}, b_{22} are the normalized normal stress deviators

$$b_{ij} = \frac{\overline{u_i u_j} - \frac{2}{3}K\delta_{ij}}{K}.$$

Using the normal stress data from the direct numerical simulation of Kim et al. (1987) produces the function shown in Fig. 1.

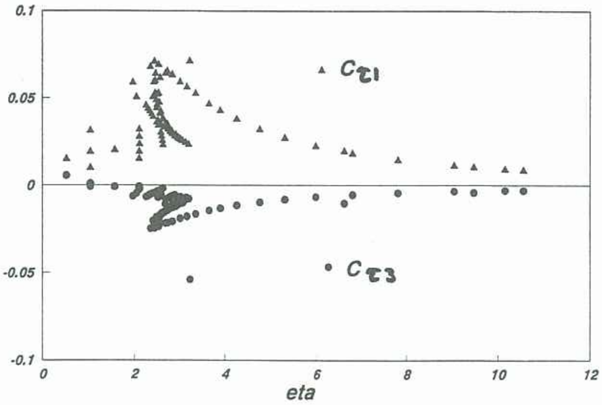

Figure 1. Variation of $C_{\tau 1}(\eta), C_{\tau 3}(\eta)$ from direct simulation data.

It is interesting to note that in the high η region, these graphs are reasonably smooth functions. The scatter in the region $\eta < 3$ is not problematic because previous investigations of channel flow show that in these regions, the normal stresses can be well predicted using constant values for the model coefficients.

The use of the η-dependent $C_{\tau 1}, C_{\tau 3}$ is illustrated by computing 2-D plane channel flow at much higher Reynolds number than the direct simulation. The $K - \varepsilon$ model of Yakhot et al. (1992) is solved along with eq. (1). The η-dependence is implemented by interpolation of the values in Fig. 1. Below $\eta \approx 3$ (the value for energy equilibrium), the constant values of Rubinstein and Barton (1990) are used. Above $\eta \approx 11$ (the limit of the direct simulation data), the coefficients are extrapolated as

$$C_{\tau 1} = 0.576\eta^{-1.795} \qquad C_{\tau 3} = -0.071\eta^{-1.4}$$

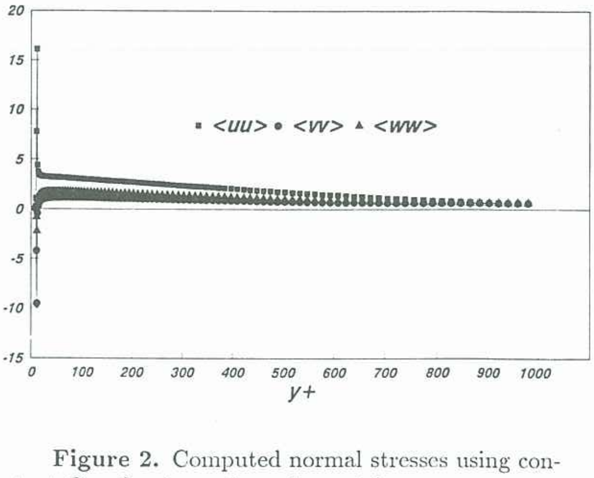

Figure 2. Computed normal stresses using constant $C_{\tau 1}, C_{\tau 3}$ for a plane channel flow.

Figures 2 and 3 demonstrate the utility of the strain dependent coefficients in eliminating the anomalous normal stress variations near the wall. Note that since the functional variation of the coefficients was determined from direct simulation data of the plane channel, the results of

Figures 2 and 3 are not true predictions, but merely verification that the model is reproducing the desired behavior. Current work in progress for the square duct exhibits the same improvement in the normal stress variation near the wall, however.

The decay of the model constants as η increases, Fig. 1, agrees qualitatively with eq. (2), although this equation gives only a fair fit to the data. Figure 4 illustrates the variation of η across the channel, indicating that the maximum values arise in the near-wall region. Turbulence in the near-wall region is in a highly strained, low Reynlds number state. The use of these direct simulation data as a near-wall correction is plausible, but does not exclude different dependence at higher Reynolds numbers.

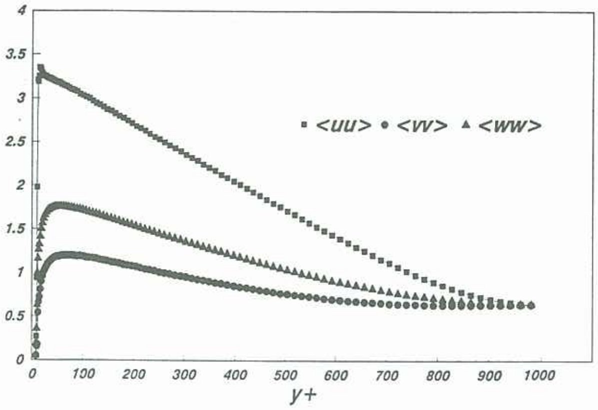

Figure 3. Computed normal stresses using $C_{\tau 1}(\eta), C_{\tau 3}(\eta)$ for a plane channel flow.

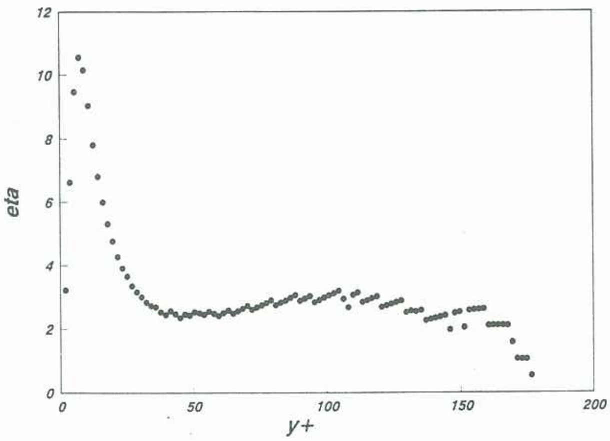

Figure 4. Variation of $\eta(y^+)$ from plane channel direct simulation data.

DISCUSSION

The current work is motivated from the desire to extend the generality and applicability of nonlinear algebraic stress models. Coupled with a $K-\varepsilon$ model, there is the potential for solving a broad spectrum of engineering problems without the complexity of full Reynolds stress transport models. Though several investigators have eschewed algebraic models for the more complex differential ones, we continue to feel the former have much to offer.

Rodi (1976) has contributed appreciably to the present status of algebraic stress models. Early on he recognized the limitation inherent in constant values for the model parameters. He proposed varying C_ν as a function of P/ε, the ratio of production to dissipation of turbulent kinetic energy. He likewise proposed a correction to the algebraic stress model for neglected transport effects (convection and diffusion). Such modifications are, however, distinct from those proposed here. For example, in simple shear flows, we write

$$\frac{P}{\varepsilon} = -\frac{S\tau}{\varepsilon} = -\frac{\tau}{K}\eta$$

where τ is the shear stress. As can be seen, except when τ/K is constant, the η variation and P/ε variation are distinct. This implies that the algebraic models can be corrected (to some extent, at least) for both transport and rapid straining effects. As shown by Yakhot et al. (1992), such models can also be corrected for some relaxation effects.

A common problem encountered in previous applications of algebraic stress models, and frequently cited as a reason to abandon thier further use, is numerical instability. When solved in conjunction with a $K-\varepsilon$ model, the stress equations enter through the production terms, and tend to make the K and ε equations source-term dominant, adversely affecting both stability and convergence. Our approach is to recast the stress equations as the sum of an isotropic and anisotropic contribution, with the isotropic part having the appearance of an eddy viscosity, but with a value of the equivalent "C_ν" different from the actual value. The anisotropic part remains as a source term but the isotropic part now contributes to the diagonal elements of the implicit matrix inversion and renders the matrix diagonally dominant. We thus do not experience any additional numerical problems relative to solution of the two-equation model.

Finally, for three-dimensional flows, the general definition of η is used, as shown just below eq. (1). However, a new problem arises, that of defining $C_{\tau 2}$. This constant does not enter for simple shear flows, but does, for example, in square duct flows. Thus, direct simulations of the plane channel are not useful for establishing its variation with η. It is of course possible to use square duct simulations for the same purpose, however, they are not currently available. For the square duct calculations in progress we have chosen to evaluate $C_{\tau 2}$ using material frame indifference, which requires $C_{\tau 2} = (C_{\tau 1} + C_{\tau 3})/2$.

ACKNOWLEDGEMENTS

The authors would like to thank Dr. Steven Orszag of Cambridge Hydrodynamics, Inc. for providing the code for plane channel calculations. This work was performed on contract NAS3-25266 with the NASA Lewis Research Center.

REFERENCES

BARTON, J M, RUBINSTEIN, R and KIRTLEY, K R (1991) Nonlinear Reynolds stress model for turbulent shear flows. AIAA Paper No. 91-0609.

CROW, S C (1968) Viscoelastic properties of fine grained incompressible turbulence. J Fluid Mech, 33, 1-20.

DEMUREN, A O and RODI, W (1984) Calculation of turbulence-driven secondary motion in non-circular ducts. J Fluid Mech, 140, 189-222.

KIM, J, MOIN, P and MOSER, R (1987) Turbulence statistics in fully developed channel flow at low Reynolds number. J Fluid Mech, 177, 133-166.

KRAICHNAN, R H (1959) J Fluid Mech, 5, 497.

LUMLEY, J L (1970) Toward a turbulent constitutive relation. J Fluid Mech, 41, 413-434.

POPE, S B (1975) A more general effective-viscosity hypothesis. J Fluid Mech, 72, 331-340.

RIVLIN, R S (1957) The relation between the flow of non-Newtonian fluids and turbulent Newtonian fluids. Q Appl Math, 15, 212-215.

RODI, W (1976) A new algebraic relation for computing the Reynolds stress. Z Angew Math Mech, 56, T219- T221.

RUBINSTEIN, R and BARTON, J M (1990) Nonlinear Reynolds stress models and the renormalization group. Phys Fluids A, 2, 1472-1476.

RUBINSTEIN, R and BARTON, J M (1991) Renormalization group analysis of anisotropic diffusion in turbulent shear flows. Phys Fluids A, 3, 415-421.

RUBINSTEIN, R and BARTON, J M (1992) Renormalization group analysis of the Reynolds stress transport equation. Phys Fluids A, 4, 1759-1766.

SPEZIALE, C G (1987) On nonlinear $K-l$ and $K-\varepsilon$ models of turbulence. J Fluid Mech, 178, 459-475.

SPEZIALE, C G (1992) private communication.

TAULBEE, D B (1991) The present status and future direction of algebraic Reynolds stress models. Workshop on Engineering Turbulence Modeling, NASA CP-10088, 101-144.

YAKHOT, V and ORSZAG, S A (1986) Renormalization group analysis of turbulence. I. Basic theory. J Sci Comput, 1, 3-51.

YAKHOT, V, THANGAM, S, GATSKI, T, ORSZAG, S A and SPEZIALE, C G (1992) Development of turbulence models for shear flows by a double expansion technique. Phys Fluids A, 4, 1510.

YOSHIZAWA, A (1984) Statistical analysis of the deviation of the Reynolds stress from an eddy-viscosity representation. Phys Fluids, 27, 1377.