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ON THE MECHANISM OF SHEAR FLOW INSTABILITY IN STRATIFIED FLUIDS AND BOUNDARY LAYERS

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Abstract.

In homogeneous and density-stratified shear flows, the conventional mechanism for instability, much invoked and discussed, is Kelvin-Helmholtz instability. There is a second mechanism, the wave-interaction mechanism, which is much more general, and is the subject of this paper. The mechanism depends on two free waves that propagate in opposite directions in a shear flow, and which may become stationary relative to each other because of the shear. If this occurs, and they have a suitably chosen phase configuration, the velocity field of each may increase the displacement of the other, and so the disturbance grows.

We show that the same mechanism is responsible for instability in a general class of symmetric but otherwise *arbitrary* velocity and density profiles, provided that the Richardson number $R_i < 1/4$ in a central region. A critical layer exists in this central region for the growing disturbance, but its role in the instability process is incidental. When $R_i > 1/4$ everywhere, the flow is stable because the free waves described above are absorbed by the critical layer, and hence are heavily damped. The necessary criteria of Rayleigh and Fjortoft for instability in homogeneous fluid are seen to provide a suitable geometry for two interacting waves.

The mechanism is also responsible for instability in other shear flows such as barotropic and baroclinic instability in rotating systems. As a further example of its generality we show that the same process applies to the growth of Tollmien-Schlichting waves in a Blasius boundary layer. These growing waves consist of an inviscid wave propagating on the vorticity gradient, interacting with a damped viscous mode in which vorticity diffuses outward from the wall in the shear flow.

1. Introduction.

In spite of the extensive work done on shear flow instability, there is little understanding of how the process works. This paper identifies a general mechanism which we suggest is the basic process behind all shear flow instabilities.

For homogeneous and stratified inviscid flows, general results about whether an arbitrarily chosen profile is or is not stable are limited to a small number of criteria that are *necessary* for instability. Specifically, these state that for the flow to be unstable we must have (e.g. Drazin & Reid 1981):

(i) the Richardson number $R_i = N^2/(dU/dz)^2$ less than 1/4 at some level in the flow (the Miles-Howard criterion);
(ii) if $N = 0$ everywhere, d^2U/dz^2 changing sign at some level in the flow (Rayleigh's criterion);
(iii) again if $N = 0$, $d^2U/dz^2(U - U(z_i)) < 0$ at some level, where z_i is the level of the inflexion point in (ii) (the Fjortoft criterion).

One would expect that an understanding of the mechanism would help to explain the significance of these criteria, and why they are necessary for instability.

2. Preliminary equations.

We consider a stably stratified inviscid fluid which has velocity and density profiles $U(z)$ and $\rho_0(z)$ respectively in the undisturbed state, where x and z are horizontal and vertical coordinates. u', w', p' and ρ' denote perturbation quantities of velocity, pressure and density from the mean values U, 0, p_0 and ρ_0. We may define a perturbation stream function ψ by

$$u' = -\frac{\partial\psi}{\partial z}, \quad w' = \frac{\partial\psi}{\partial x}. \qquad (1)$$

If we make the Boussinesq approximation for convenience and eliminate all other variables in favour of ψ, we obtain

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\psi + N^2\frac{\partial^2\psi}{\partial x^2} - \frac{d^2U}{dz^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial\psi}{\partial x} = 0, \qquad (2)$$

where N is the buoyancy frequency, defined by $N^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dz}$. As is standard practice for instability studies, we look for disturbances in the form of normal modes, namely

$$\psi = \hat{\psi}(z)\, e^{ik(x-ct)} \qquad (3)$$

and obtain the equation for $\hat{\psi}$

$$L(\hat{\psi}) \equiv \frac{d^2\hat{\psi}}{dz^2} + \left(\frac{N^2(z)}{(U(z)-c)^2} - \frac{\frac{d^2U}{dz^2}}{(U-c)} - k^2\right)\hat{\psi} = 0. \qquad (4)$$

For given velocity and density profiles, solutions of this equation give eigenvalues for c and eigenfunctions for $\hat{\psi}$. If the former has a complex value, the flow has a growing mode and is deemed to be unstable. For any given solution, the vertical displacement η is given by

$$w' = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta, \quad \text{so that } \eta = \hat{\psi}/(U-c). \qquad (5)$$

3. A simple prototype: the case of two vorticity interfaces.

We next consider a system with two vorticity interfaces at $z = \pm d$ in a homogeneous fluid, where the fluid velocity varies linearly between them and is constant ($= \pm U$) for $|z| > d$. The vorticity increments at the interfaces, $\Delta\zeta = \pm U/d$, are equal and opposite. The solution (Rayleigh, 1896) is given by

$$\hat{\psi} = e^{-k|z-d|-i\vartheta} - e^{-k|z+d|+i\vartheta}, \qquad (6)$$

$$\left(\frac{c}{U}\right)^2 = \frac{(1-2kd)^2 - e^{-4kd}}{(2kd)^2}, \qquad (7)$$

and ϑ depends on c. The disturbance consists of two free waves with a specified phase ϑ, and c dependent on the interaction. If $0 < kd < k_s d \cong 0.64$, c is purely imaginary with $c = ic_i$ and the flow is unstable; ϑ is then given by

$$\tan 2\vartheta = -\frac{2kdc_i/U}{1-2kd}, \qquad (8)$$

and lies in the range $0 < \vartheta < \pi/2$. The disturbance is stationary (in this coordinate system) and grows with time, but the relative phase of the interface displacements depends on ϑ. The motion may be regarded as a pair of waves propagating on each vorticity interface, each being affected by the velocity field of the other. With the stationary phase configuration of (6), the velocity field of each wave may be expressed as the sum of a component which is in phase with the displacement of the other wave (meaning that the velocity and displacement maxima coincide), plus a second component which is out of phase with it by $\pi/2$. The first component will act to increase the amplitude of the displacement of the other wave, so that the disturbance grows with time. The finite bandwidth of the instability, $0 < kd < k_s d$, occurs because the second component of one wave will affect the speed of the other, and may act to keep it stationary relative to the first wave. This allows the first component to act.

4. Generalisation to arbitrary symmetric profiles.

The above mechanism is also valid for a wide class of velocity and density profiles. We can demonstrate this by describing the properties of a specific type of general stratified shear flow. The development follows a framework outlined by Cairns (1979) and Drazin (1989). We consider a system as shown in Figure 1, where the flow is confined between horizontal rigid boundaries at z_1, z_2 (which may be at infinity), and we may identify three flow regions, as follows. There is an uppermost region $l_2 < z < z_2$ in which the flow is stable, in the sense that it

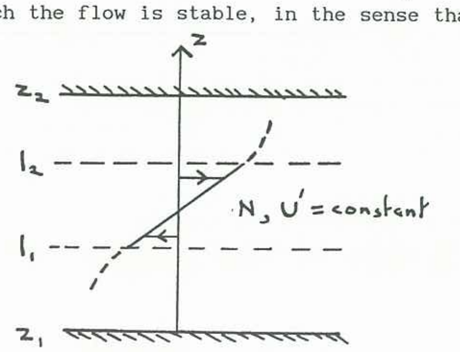

Figure 1. Configuration for the general case discussed in section 4.

fails to satisfy any of the necessary criteria for instability in section 1 (for example $R_i > 1/4$, or $d^2U/dz^2 \neq 0$ if $N = 0$, at all levels). The flow in the lowest region $z_1 < z < l_1$ is similarly stable, and these two regions are separated by a central region $l_1 < z < l_2$ in which N and dU/dz are constant. To investigate the stability of such a flow, we look for eigensolutions ψ of (4) which satisfy the boundary conditions

$$\hat{\psi}(z_1) = 0 = \hat{\psi}_2(z_2), \qquad (9)$$

and for which the flow is unstable if the corresponding eigenvalue c is complex.

For given values of k and c, we may define two solutions $\hat{\psi}_1(z,k,c)$ and $\hat{\psi}_2(z,k,c)$ of (4) such that $\hat{\psi}_1$ satisfies the lower boundary condition, and $\hat{\psi}_2$ the upper. Specifically, this means

$$L(\hat{\psi}_1) = 0, \quad \text{with } \hat{\psi}_1(z_1) = 0,$$
$$L(\hat{\psi}_2) = 0, \quad \text{with } \hat{\psi}_2(z_2) = 0. \qquad (10)$$

This determines each of $\hat{\psi}_1$ and $\hat{\psi}_2$, apart from a multiplicative factor. For example, for given k and c, one may obtain a representative form for $\hat{\psi}_1$ by integrating (4) from $z = z_1$ with some arbitrarily chosen value of $d\hat{\psi}_1/dz$. For either of $\hat{\psi}_1$ or $\hat{\psi}_2$ to be an eigensolution, it must satisfy *both* boundary conditions, which means $\hat{\psi}_1 = \text{constant}.\hat{\psi}_2$, or equivalently

$$W(\hat{\psi}_1,\hat{\psi}_2) = \hat{\psi}_1\hat{\psi}_2' - \hat{\psi}_1'\hat{\psi}_2 = 0, \qquad (11)$$

where ψ' denotes $d\psi/dz$.

In the central region where the Richardson number is constant, we consider three specific situations with R_i having different values or ranges, the first of which is zero.

<u>*Case 1*</u>. *$R_i = 0$ in the central region.*

Since $N = 0$ in this region, $\hat{\psi}_1$ and $\hat{\psi}_2$ here have the form

$$\hat{\psi}_1 = D_1(k,c).e^{k(z-l_1)} + B_1(k,c).e^{-k(z-l_1)},$$
$$\hat{\psi}_2 = B_2(k,c).e^{k(z-l_2)} + D_2(k,c).e^{-k(z-l_2)}, \quad l_1 < z < l_2. \qquad (12)$$

The functions D_1 and B_1 depend on unspecified details of the flow in the lower region alone, and similarly for D_2 and B_2 for the upper region. However, if z_1 and l_1 are fixed and the central region is broadened so that l_2 becomes large, the effects of the upper region on the lower are removed, and we have

$$l_2 \to \infty, \qquad \hat{\psi}_1 \to B_1(k,c).e^{-k(z-l_1)}, \quad z > l_1,$$
$$D_1(k,c) = 0. \qquad (13)$$

D_1 is independent of the upper region, and $D_1 = 0$ gives the dispersion relation for the lower region in isolation. Similarly, if z_2 and l_2 are fixed and $l_1 \to -\infty$, we have

$$l_1 \to -\infty, \qquad \hat{\psi}_2 \to B_2(k,c).e^{k(z-l_2)}, \quad z < l_2,$$
$$D_2(k,c) = 0. \qquad (14)$$

Hence, when the upper and lower regions become isolated independent waveguides, the dispersion relations for each are that D_1 and D_2 are zero. For the complete system, however, if $\hat{\psi}_1$ or $\hat{\psi}_2$ is to be a solution they must satisfy (11), which gives the dispersion relation

$$D_1 D_2 = e^{-2k(l_2 - l_1)} B_1 B_2 . \qquad (15)$$

Solutions for c are eigenvalues for the complete system, and complex roots of (15) imply instability. In the central region $l_1 < z < l_2$, the solution takes the form of the sum of two free waves but with c determined by (15), as for the prototype case in section 3.

We next consider flows where the velocity profile is anti-symmetric about a particular level (taken as z = 0), and the buoyancy is symmetric, so that

$$U(-z) = - U(z), \quad N^2(-z) = N^2(z) . \qquad (16)$$

With these relations one may readily show that solutions of (4) satisfy

$$\psi(z,k,c) = \psi(-z,k,-c), \qquad (17)$$

and hence

$$D_2(k,c) = D_1(k,-c), \quad B_2(k,c) = B_1(k,-c). \qquad (18)$$

If the lower region has a finite number of modes (n say), we may write

$$D_1(k,c) = \prod_{j=1}^{n} (c_j(k)-c) , \qquad (19)$$

where c_i is the speed of the jth free wave mode. More generally, for one particular mode, we may write

$$D_1(k,c) = (c_j(k)-c)d_1(k,c). \qquad (20)$$

(15) may then be written in the form

$$(c_j^2-c^2)d_1(k,c)d_1(k,-c) = \varepsilon^2 B_1(k,c)B_1(k,-c). \qquad (21)$$

Suppose next that $c_j(k) = 0$, so that two free wave modes are stationary in this reference frame, for this k value. Then (4.21) gives

$$c^2 = - \varepsilon^2 \frac{B_1(k,c)B_1(k,-c)}{d_1(k,c)d_1(k,-c)} , \qquad (22)$$

and if ε is sufficiently small, this must have roots $c = \pm ic_i$, where

$$c_i = \varepsilon \left| \frac{B_1(k,c)}{d_1(k,c)} \right| , \qquad (23)$$

and hence the flow must be unstable. As for the prototype example of section 3, there will be a finite bandwidth of unstable wavenumbers centred (approximately) on the criterion for exact resonance. Hence, the essence of the instability process depends on the mutual interaction between two free waves that propagate in opposite directions.

Case 2: $0 < R_i < 1/4$ *in the central region.*

For this case we may follow the same procedure as in Case 1 to obtain essentially the same result, although the details are more complex. If we define

$$\xi = k(z-c), \quad \xi_i = k(l_i-c), \quad i = 1,2, \qquad (24)$$

then for $l_1 < z < l_2$ we may write $\hat{\psi} = \xi^{1/2}\phi(\xi)$, where ϕ satisfies

$$\frac{d^2\phi}{d\xi^2} + \frac{1}{\xi}\frac{d\phi}{d\xi} - \left(1 + \frac{\nu^2}{\xi^2}\right)\phi = 0, \quad \nu = \left(\tfrac{1}{4} - R_i\right)^{1/2} . \qquad (25)$$

The Hankel functions $H_\nu^{(1)}(i\xi)$, $H_\nu^{(2)}(i\xi)$ are solutions of (25), and these may be used to construct solutions corresponding to (12), which on the real axis of the ξ-plane have the form

$$\hat{\psi}_1 = \left(\frac{\xi}{\xi_1}\right)^{1/2}\left(D_1 \frac{H_\nu^{(2)}(i\xi)}{H_\nu^{(2)}(i\xi_1)} + B_1 \frac{H_\nu^{(1)}(i\xi)}{H_\nu^{(1)}(i\xi_1)}\right), \quad \xi_1 < \xi < \xi_2,$$

$$\hat{\psi}_2 = \left(\frac{\xi}{\xi_2}\right)^{1/2}\left(D_2 \frac{H_\nu^{(1)}(i\xi)}{H_\nu^{(1)}(i\xi_2)} + B_2 \frac{H_\nu^{(2)}(i\xi)}{H_\nu^{(2)}(i\xi_2)}\right), \qquad (26)$$

where $c = c_r + ic_i$. As for Case 1, $D_1 = 0$ gives the dispersion relation for waves in the lower region, and $D_2 = 0$ for the upper region. Substituting the expressions (26) into (11) then gives the dispersion relation

$$D_1 D_2 - B_1 B_2 \frac{H_\nu^{(2)}(i\xi_1)\, H_\nu^{(1)}(i\xi_2)}{H_\nu^{(1)}(i\xi_1)\, H_\nu^{(2)}(i\xi_2)} = 0. \qquad (27)$$

If we again restrict consideration to the symmetric profiles (16), the solutions satisfy (4.9), and D_1, B_1 in (26) satisfy (18). In the case where a free mode of the upper or lower region is stationary, so that $c_j = 0$ in (20), then

$$c^2 = - \frac{B_1(k,c)B_1(k,-c)}{d_1(k,c)d_1(k,-c)} \left| \frac{H_\nu^{(1)}(i\xi_2)}{H_\nu^{(2)}(i\xi_2)} \right|^2 , \qquad (28)$$

and the arguments for Case 1 again apply. In particular, the flow must be unstable if $2kl_2$ is sufficiently large, and this is due to the mutual interaction of two free modes.

Case 3. $R_i > 1/4$ *in the central region.*

Here we know from the Miles-Howard theorem that the flow is stable, and the interest centres on the way in which the mechanism of instability of Cases 1 and 2 breaks down. In this case, from (24), (25), the general solution to (4) has the form

$$\psi = \xi^{1/2}(A_1 I_{i\mu}(\xi) + A_2 I_{-i\mu}(\xi)) , \qquad (29)$$

where $\mu = (R_i - 1/4)^{1/2}$, and A_1, A_2 are constants. Whereas $H_\nu^{(1)}(i\xi)$ and $H_\nu^{(2)}(i\xi)$ have monotonic near-exponential behaviour for ξ real, $I_{\pm i\mu}(\xi)$ are oscillatory, and the shear is weak enough to permit wave propagation in the region. If waves propagating on the upper and lower regions are to be stationary relative to each other, so that they can interact as for Cases 1 and 2, they must have a critical level in the central region, $l_1 < z < l_2$. A single wave on the upper or lower region that has a critical layer in the central region will have energy propagation toward the critical layer, with a decrease in amplitude of $e^{-\mu\pi}$ and a phase change of π/2 across it (Booker & Bretherton 1967). Such a wave embodies a constant flux of momentum $\rho_o \overline{u'w'}$ toward (or away from, depending on sign) the critical layer, and this flux is discontinuous across it, decreasing in magnitude by the factor $e^{-2\mu\pi}$. There is no reflection from the critical layer, and for this system to be steady, the incident wave must be maintained by some forcing due to, for example, flow over sinusoidal topography. In the absence of such forcing, there are no neutral wave modes on the upper and lower "waveguides" because the waves approaching the critical layer are not reflected. If such a system with two waves of equal amplitude on each side of the central region is to be unstable, therefore, the waves must be able to force each other across the critical layer at sufficient strength to overcome the loss due to critical layer absorbtion. The $e^{-\mu\pi}$ factor weakens these effects, so that they are not sufficient for this, and consequently the flow is stable. Hence, we attribute the stability of the flows where $R_i > 1/4$ everywhere to the *non-existence* of the discrete neutral modes, whose mutual forcing causes instability when $R_i < 1/4$.

Note that the necessary criteria of Rayleigh and Fjortoft when N = 0 permit two suitable wave guides to exist, in which two waves can become stationary.

5. The stability of a Blasius boundary-layer over a flat plate.

The instability mechanism also applies to viscous flows, and we demonstrate this by considering the case of a Blasius boundary-layer velocity profile in homogeneous fluid as shown in Figure 2. Only a brief outline can be given here, and details are given in Baines & Mitsudera (1992). The equations governing small disturbances are (1-4) with the viscous terms added, and if ζ is the perturbation vorticity, the vorticity equation is then

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\zeta - w'\frac{d^2U}{dz^2} = \nu\nabla^2\zeta \ , \ \zeta = \nabla^2\psi \ . \qquad (30)$$

This gives the Orr-Sommerfeld equation for $\hat{\psi}$, with the boundary conditions

$$\psi,\ \psi' = 0,\ z = 0, \qquad \psi,\ \psi' \to 0 \text{ as } z \to \infty. \qquad (31)$$

For present purposes we may conveniently approximate the Blasius velocity profile by the piecewise linear profile shown in Figure 2, where the shear is uniform up to a height $d = 3.25(\nu X/U_0)^{1/2}$. The Reynolds number $R_e = U_0 d/\nu$.

The solution of this mathematical problem for the eigenvalues c and the eigenfunctions ψ goes back to Tollmien and Schlichting and is well-described in several texts (e.g. Schlichting 1968, Drazin & Reid 1981), and the stability properties of the system are well known. Our purpose here is to describe the mechanics of this process. To do this, we examine the possible free modes that may exist in separate parts of the system in isolation. There are two such free modes that are relevant here, an inviscid mode, and a viscous mode.

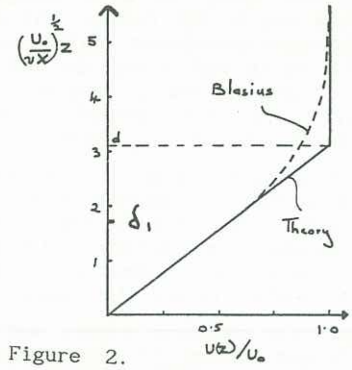

Figure 2.

The inviscid mode.

This is a free interfacial mode propagating on the vorticity interface as in section 2, but subject to the lower boundary condition. In terms of ψ it has the form

$$\hat{\psi} = \frac{\sinh kz}{\sinh kd}\ , \qquad 0 \le z \le d\ ,$$
$$= e^{-k(z-d)}\ , \qquad z \ge d\ , \qquad (32)$$

and with $\alpha = kd$ the eigenvalues are

$$\frac{c}{U_0} = 1 - \frac{\tanh \alpha}{\alpha(1 + \tanh \alpha)}\ . \qquad (33)$$

The viscous mode.

The main part of the viscous mode is located much closer to the wall than is that for the inviscid mode. For this mode we assume $\bar{u}'' = 0$, in (30), with the boundary conditions (31). The solution for the vorticity $\zeta = \Phi e^{ik(x-ct)}$ has the Airy function solution

$$\Phi = \mathrm{Ai}(Z) = \frac{1}{\pi}\left(\frac{Z}{3}\right)^{1/2} K_{1/3}\left(\frac{2}{3}Z^{3/2}e^{i\pi}\right), \qquad (34)$$

where K is the modified Bessel function and

$$Z = -(\alpha R_e)^{1/3}\left[i\alpha\left(\frac{z}{d} - \frac{c}{U_0}\right) + \alpha^2/(\alpha R_e)^{2/3}\right]. \qquad (35)$$

The relevant eigenvalue is

$$(c_r + ic_i)/U_0 = \frac{(3.67 - 2.16i)}{(\alpha R_e)^{1/3}}\ , \qquad (36)$$

giving a decaying mode. This mode embodies a balance between horizontal advection of vorticity by a uniform shear, and upward diffusion.

The criterion for resonance may be obtained by equating c_r for the viscous mode with speed c for the inviscid mode, to give

$$R_e = \frac{49.43}{\alpha\left[1 - \frac{1}{2\alpha}(1 - e^{-2\alpha})\right]^3}\ . \qquad (37)$$

This relation is shown plotted in Figure as the curve AB, where it has been superimposed on the computed regions for instability in the Reynolds number - wavenumber diagram given in Schlichting (1968). The fact that the resonance curve passes approximately through the centre of the unstable region is convincing evidence that this instability obtained by solving the *full* eigenvalue problem is caused by interaction between the two free modes. Instability arises where the interaction mechanism is strong enough to overcome the inherent damping of the free viscous mode, and this occurs provided that R_e is large enough.

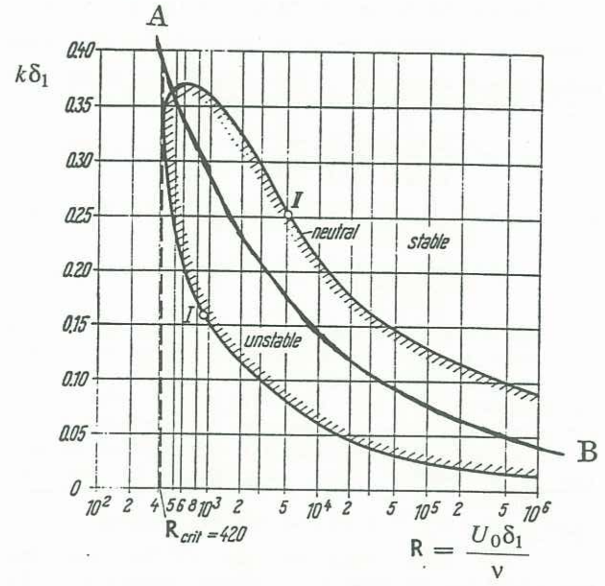

Figure 3. The regions of instability of a Blasius boundary-layer obtained from Schlichting (1968), with the resonance curve AB of equation (37) added.

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