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A DYNAMICAL SYSTEMS MODEL FOR THE AXISYMMETRIC JET MIXING LAYER

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ABSTRACT

The near field jet mixing layer is modeled in a manner similar to that applied by Aubry et al (1988) to the near wall region of the turbulent boundary layer. In this work the instantaneous velocity field is expanded in terms of the empirical eigenfunctions obtained by Glauser and George (1987). These eigenfunctions were extracted from the measured cross-spectral tensor by application of the proper orthogonal decomposition theorem (POD) suggested by Lumley (1967). Galerkin projection is then applied to the Navier Stokes equations in conjunction with this representation, resulting in low-dimensional sets of ordinary differential equations. The methods of dynamical systems theory are then used to analyze these equations. This work consists of an attempt to utilize the dynamical systems model to further our understanding of the transfer of turbulent energy between various azimuthal modes and streamwise wavenumbers and relate this to the turbulence production phenomena in the jet mixing layer. With this model the sequence by which the various modes contribute in time can be examined.

INTRODUCTION

In recent years, two separate developments have altered the basic statistical framework of turbulence. There is an abundance of experimental evidence implying the existence of coherent structures and, on the theoretical side, recent applications of dynamical systems theory to turbulence suggest that such flows reside on relatively low dimensional manifolds or attractors(v. Aubry et al (1988) and references therein).

Aubry et al (1988) appear to be among the first to link low dimensional chaotic dynamics to a turbulent open flow system. They used the POD to provide basis functions to obtain low dimensional sets of ordinary differential equations from the Navier–Stokes equations. Used in conjunction with Galerkin projection, the POD yields an optimal set of basis functions in the sense that the resulting truncated system of ordinary differential equations captures the maximum amount of kinetic energy among all possible truncations in the same order. Sirovich and Rodriguez (1987) have shown, for the Ginzburg–Landau system, that these basis functions are fairly robust and can be used over a wide range of the bifurcation parameter. The basis functions that Aubry et al (1988) used were those obtained experimentally by Herzog (1986). The equations that they derived, which exhibit intermittent behavior, where then analyzed using dynamical systems theory. The results to date show consistency between the behavior of these equations and events seen in experimental work.

The axisymmetric jet is a good candidate for a similar approach because the series converges quickly. This was demonstrated by Glauser et al (1987), where the instantaneous signals were almost completely reconstructed with only 3 terms from the expansion. In this work the instantaneous velocity field of the axisymmetric jet mixing layer is expanded in terms of the empirical eigenfunctions obtained by Glauser and George (1987). These eigenfunctions were extracted from the measured cross spectral tensor by application of the POD. In the streamwise directions the flow is approximately homogeneous, so as a first step, the Fourier modes were used. In the azimuthal direction the flow is periodic, hence the use of the Fourier modes in this direction as well. In the radial direction the flow is strongly inhomogeneous so that the eigenfunctions obtained from applying the POD were utilized. Galerkin projection is then used in conjunction with the POD to obtain a truncated system of ordinary differential equations. The modes neglected in the truncation are accounted for by a Heisenberg model. This work appears to be the first attempt at utilizing low dimensional dynamics in conjunction with Galerkin projection and the POD to examine the temporal dynamics of coherent structures in a high Reynolds number axisymmetric jet free shear layer.

1. PROPER ORTHOGONAL DECOMPOSITION

In 1967 Lumley suggested that the coherent structure should be that structure which has the largest mean square projection on the velocity field. This process involves maximizing the mean square energy via the calculus of variations and leads to the following integral eigenvalue problem.

$$\lambda\phi_i(\tilde{x}) = \int R_{ij}(\tilde{x},\tilde{x}')\phi_j(\tilde{x}')d\tilde{x}' \tag{1.1}$$

The symmetric kernel of this Fredholm integral equation is the two-point correlation tenser R_{ij} defined by

$$R_{ij}(\tilde{x},\tilde{x}') = \overline{u_i(\tilde{x})u_j(\tilde{x}')}, \tag{1.2}$$

where $\tilde{\phi}$ is the candidate structure and $\tilde{x}$ and $\tilde{x}'$ represent different spatial points in the inhomogeneous directions and different times if the flow is non-stationary.

From the Hilbert-Schmidt theory it can be shown that the solution of a Fredholm integral equation of the first kind for a symmetric kernel is a discrete set, hence equation (1.1) can be written as

$$\lambda^n\phi_i^n(\tilde{x}) = \int R_{ij}(\tilde{x},\tilde{x}')\phi_j^n(\tilde{x}')d\tilde{x}' \tag{1.3}$$

The eigenfunctions of the Fredholm equation are orthogonal over the interval and

$$\int \phi_i^n(\tilde{x})\phi_i^m(\tilde{x})d\tilde{x} = \delta_{nm} \tag{1.4}$$

for normalized eigenfunctions. The eigenvalues of the Fredholm equation with a real symmetric kernel are all real and uncorrelated

$$\overline{a^n a^m} = \lambda^n \delta_{nm} \tag{1.5}$$

and the fluctuating random field $\tilde{u}_i$ can be reconstructed from the eigenfunctions in the following way

$$\tilde{u}_i(\tilde{x}) = \sum_{n=0}^{\infty} a^n \phi_i^n(\tilde{x}). \tag{1.6}$$

The random coefficients are calculated from

$$a^n = \int \tilde{u}_i(\tilde{x})\phi_i^{n*}(\tilde{x})d(\tilde{x}) \tag{1.7}$$

where the ϕ_i^n are the eigenfunctions obtained from equation (1.3). The turbulent kinetic energy is the sum over n of the eigenvalues λ^n, and each structure makes an independent contribution to the kinetic energy and Reynolds stress.

If the random field is homogeneous or periodic in one or more directions or stationary in time, the eigenfunctions become Fourier modes, so that the POD reduces to the harmonic orthogonal decomposition in these directions. In the jet studied the flow is periodic in the azimuthal direction and almost homogeneous in the streamwise direction. We will not transform over time in this case because we are keeping time in the random coefficients. Under these conditions the spectral tensor may be defined by

$$S_{ij}(r,r',m,k_1) = \int R_{ij}(r,r',\theta,x)e^{-2\pi i(k_1x+m\theta)}dxd\theta \tag{1.8}$$

so that equation (1.3) becomes

$$\lambda^n(m,k_1)\phi_i^n(r,m,k_1) = \int_\Omega S_{ij}(r,r',m,k_1)\phi_j^n(r',m,k_1)dr'. \tag{1.9}$$

In the above expressions r and r' represent different spatial locations in the radial direction (the inhomogeneous direction), x and θ are the separations in the streamwise and azimuthal directions respectively, k_1 is the streamwise wavenumber and m is the azimuthal mode number. This equation is then solved numerically using the measured values of $S_{ij}(r,r',m,k_1)$ obtained by Glauser and George (1987). It should be noted that in this case the eigenvalues and eigenfunctions are now a function of azimuthal mode number m and streamwise wavenumber k_1.

2. THE MOMENTUM EQUATION

The Navier–Stokes equations for an incompressible flow in cylindrical coordinates, after application of Reynold's decomposition and substitution of a relationship between the divergence of Reynold's stress and the mean pressure and velocity , can be written as:

$$\frac{\partial}{\partial t}\tilde{u}_i + M\tilde{u}_i + (-\frac{\tilde{u}_\theta^2}{r} + \overline{U}_z\frac{\partial \tilde{u}_r}{\partial z})\delta_{ir} + (\frac{\tilde{u}_r\tilde{u}_\theta}{r} + \overline{U}_z\frac{\partial \tilde{u}_\theta}{\partial z})\delta_{i\theta}$$

$$+(\overline{U}_z\frac{\partial \tilde{u}_z}{\partial z} + \tilde{u}_r\frac{\partial \overline{U}_z}{\partial r})\delta_{iz} + \overline{M\tilde{u}_i} - \frac{\overline{\tilde{u}_\theta^2}}{r}\delta_{ir} + \frac{\overline{\tilde{u}_r\tilde{u}_\theta}}{r}\delta_{i\theta}$$

$$= -\frac{r(\delta_{ir}+\delta_{iz})+\delta_{i\theta}}{r\rho}\frac{\partial \tilde{p}}{\partial i}$$

$$+\nu[\nabla^2\tilde{u}_i - (\frac{\tilde{u}_r}{r^2} + \frac{2}{r^2}\frac{\partial \tilde{u}_\theta}{\partial \theta})\delta_{ir} + (-\frac{\tilde{u}_\theta}{r^2} + \frac{2}{r^2}\frac{\partial \tilde{u}_r}{\partial \theta})\delta_{i\theta}]. \tag{2.1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},$$

$M = \tilde{u}_r\frac{\partial}{\partial r} + \frac{\tilde{u}_\theta}{r}\frac{\partial}{\partial \theta} + \tilde{u}_z\frac{\partial}{\partial z}$, $i = r,\theta,z$, δ is the Kronecker delta, $\tilde{p}$ is the fluctuating pressure , $\tilde{u}_r$, $\tilde{u}_\theta$ and $\tilde{u}_z$ are the fluctuating velocities in the radial, azimuthal and streamwise direction respectively, and $\overline{U}_z$ is the mean streamwise velocity. The overbar denotes an average, which in this case is defined as a spatial average over the homogeneous and periodic directions (v. Aubry et. al. (1988)). In the above equation it is assumed that ρ and μ are constant and that the body forces are zero.

One change is made to the above equation before we perform a Fourier Transform and the Galerkin projection on it. This involves deriving a relationship between the mean streamwise velocity and the fluctuating velocity and pressure from the Navier-Stokes equations. This change is needed because the actual measured mean velocity profile will be incorrect for the severely truncated system of equations to be studied. This relationship, which expresses the mean velocity $\overline{U}_z$ in terms of the Reynolds stress $\overline{\tilde{u}_r\tilde{u}_z}$ can be shown to be (v. Zheng (1990))

$$\overline{U}_z = \frac{1}{\nu}\int \overline{\tilde{u}_r\tilde{u}_z}dr \tag{2.2}$$

where, as a first step, we have neglected the terms which come from the pressure (i.e., an isobaric jet). Equation (2.2) is then used to eliminate $\overline{U}_z$ from equation (2.1). In the axisymmetric jet this gives some feedback to the system of equations as the fluctuation changes. It should be noted that this feedback increases as the Reynolds stress gets stronger. This is similar to the relationship derived by Aubry et al. (1988) for the turbulent boundary layer.

3. GALERKIN PROJECTION AND ENERGY TRANSFER MODEL

The Galerkin method is well known and has been used extensively to study turbulence and the instability of various fluid flows (v. Lin et al (1987) and references therein). The essential idea of the method is to expand the dependent variables in terms of a finite series of independent basis functions. The basis functions form a complete basis for the relevant class of functions and they satisfy the relevant boundary conditions. In this work we use a Galerkin projection in conjunction with the POD (to supply the basis functions) which minimizes the error due to the truncation and yields a set of ordinary differential equations for the coefficients (v. equation 1.7).

The Galerkin projection is performed on the Fourier Transform (in the azimuthal and streamwise directions) of the Navier Stokes equations so that is is useful to define the following equations:

$$\tilde{u}_i(z,\theta,r,t) = \sum_m \int_{-\infty}^{\infty} e^{-2\pi j(k_1z+m\theta)}\hat{u}_i(k_1,m,r,t)dk_1, \tag{3.1}$$

and

$$\hat{u}_i(k_1,m,r,t) = \int\int_{-\infty}^{\infty} e^{2\pi j(k_1z+m\theta)}\tilde{u}_i(z,\theta,r,t)dzd\theta. \tag{3.2}$$

We expand $\hat{u}_i(k_1,m,r,t)$ in terms of the coefficients and eigenfunctions in the following manner(v. equation (1.6)),

$$\hat{u}_i(k_1,m,r,t) = \sum_n a_{k_1,m}^n(t)\phi_{i,k_1,m}^n(r). \tag{3.3}$$

We then substitute these equations into the Fourier Transform of the Navier-Stokes equations and perform the Galerkin projection. This is written as

$$(\phi^l, N) = \int_0^R N_i(t,r)\phi_i^{l*}(r)dr = 0 \qquad (3.4)$$

where $N_i(t,r)$ represents the Fourier Transform of the Navier Stokes equations.

Finally, after performing the above and utilizing the orthogonality condition (v. equation (1.4)) we obtain a set of ordinary differential equations for the coefficients which follow:

$$\frac{da^l_{k_1,m}}{dt} = \nu \sum_n a^n_{k_1,m} \qquad \text{(linear term)}$$

$$\{-4\pi^2 k_1^2 \delta_{nl} + \int_0^R [(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4\pi^2 m^2}{r^2})\phi^n_{i,k_1,m}$$

$$-(\frac{1}{r^2}\phi^n_{r,k_1,m} + \frac{4\pi j m}{r^2}\phi^n_{\theta,k_1,m})\delta_{ir}$$

$$+(-\frac{1}{r^2}\phi^n_{\theta,k_1,m} + \frac{4\pi j m}{r^2}\phi^n_{r,k_1,m})\delta_{i\theta}] \cdot \phi^{l*}_{i,k_1,m} dr\}$$

$$-\sum_{k_1' m'}\sum_{pq} a^p_{k_1',m'} a^q_{k_1-k_1',m-m'} \qquad \text{(quadratic term)}$$

$$\{\int_0^R [\phi^p_{r,k_1',m'} \frac{d}{dr}\phi^q_{i,k_1-k_1',m-m'} + \frac{2\pi j(m-m')}{r}$$

$$\phi^p_{\theta,k_1',m'}\phi^q_{i,k_1-k_1',m-m'} + 2\pi j(k_1 - k_1')\phi^p_{z,k_1',m'}\phi^q_{i,k_1-k_1',m-m'}$$

$$-(\frac{1}{r}\phi^p_{\theta,k_1',m'}\phi^q_{\theta,k_1-k_1',m-m'})\delta_{ir}$$

$$+(\frac{1}{r}\phi^p_{r,k_1',m'}\phi^q_{\theta,k_1-k_1',m-m'})\delta_{i\theta}] \cdot \phi^{l*}_{i,k_1,m} dr\}$$

$$-\frac{1}{\nu}\sum_{k_1' m'}\sum_{pqr} a^p_{k_1',m'} a^{q*}_{k_1',m'} a^r_{k_1,m} \qquad \text{(cubic term)}$$

$$\{\int_0^R [(\int_0^{R'} \phi^p_{r,k_1',m'}\phi^{q*}_{z,k_1',m'} dr')\phi^r_{i,k_1,m} 2\pi j(k_1)$$

$$+\phi^p_{r,k_1',m'}\phi^{q*}_{z,k_1',m'}\phi^r_{r,k_1,m}\delta_{iz}] \cdot \phi^{l*}_{i,k_1,m} dr\}$$

$$-\int_0^R \frac{r(\delta_{ir} + \delta_{iz}) + \delta_{i\theta}}{\rho r}\frac{\partial \hat{p}_{k_1,m}}{\partial i} \cdot \phi^{l*}_{i,k_1,m} dr\} \qquad (3.5)$$

where $i = r, \theta, z$ and δ is the Kronecker delta. In our ordinary differential equation

$$A\frac{da^l}{dt} = Bl + Cq + Dc \qquad (3.6)$$

there are four parts. The term on the left hand side is the time derivative of the coefficient a. The first term on the right hand side (RHS) is a linear term that come from diffusion term of the Navier-Stokes equations. The second term on the RHS is a quadratic nonlinear term that is a consequence of the fluctuation–fluctuation interactions and exhibits the energy transfer between the different eigenfunctions(from the POD) and the Fourier modes. The last term on the RHS is a cubic term that is a direct result of the mean velocity–fluctuation interaction. A, B, C and D are all matrices and A is an an identity matrix because of the orthogonality of the eigenfunctions. It should be noted that the pressure term in equation (3.5) will vanish if the integration covers the whole domain. In the jet shear layer we cover most of the domain in our integration (unlike Aubry et al. (1988) who studied the near wall region and not the whole domain of the turbulent boundary layer), hence we will neglect this pressure term.

When we truncate at some cutoff point we need to account for the energy transfer between the included and neglected modes. The effect of the neglected modes will be accounted for by utilizing a Heisenberg model(v. Aubry et al. (1988)). The assumption is that the neglected modes withdraw energy from the modes that our kept, as if a certain turbulence viscosity were present.

The equation for ν_T, our turbulence viscosity, can be shown to be (v. Zheng (1990))

$$\nu_T = \frac{\alpha \sum_{k_1,m,n} \lambda^n_{k_1,m}}{R \sum_{k_1,m,n} \lambda^n_{k_1,m} \overline{S}} \qquad (3.7)$$

where

$$\overline{S} = \overline{\int_0^R [\frac{d\phi^n_{i,k_1,m}}{dr}\frac{d\phi^{n*}_{i,k_1,m}}{dr} - \frac{1}{r^2}4\pi^2 m^2 \phi^n_{i,k_1,m}\phi^{n*}_{i,k_1,m}]dr - 4\pi^2 k_1^2}$$

and α is a dimensionless parameter. We now substitute ν_T into our ordinary differential equations. The equations then have the form

$$A\frac{da^l}{dt} = B(1 + \nu_T/\nu)l + Cq + Dc. \qquad (3.8)$$

The effect of the energy transfer model is to introduce the parameter α which becomes the bifurcation parameter in our system of ODE's. The larger α the more energy that the neglected modes take from our system so that the system should be stable. As α decreases less energy is extracted so that we expect our system to become unstable.

4. TRUNCATIONS AND DISCUSSION

At the present time various truncations are being explored. A basic guide in the selection of which terms to keep is to retain a minimum number of terms but yet keep as much energy in the system as is necessary to retain the essential dynamics of the flow phenomena (v. Aubry et al (1988)). Figure 1 is a plot of the dominant eigenvalue obtained from the application of the POD, plotted versus streamwise wavenumber and azimuthal mode number. This plot indicates that there is an exchange of energy between streamwise wavenumbers and azimuthal mode numbers, and in particular, that there is a maximum energy path (remember that the eigenvalues are energy integrated across the jet shear layer) between the two peaks. The peak in the wavenumber direction corresponds to the Strouhal frequency of the jet. The peak in the azimuthal mode number direction is approximately at mode 6. Several azimuthal modes around the peak are kept as well as several around the peak in the streamwise wavenumber direction in order to retain the essential dynamics. For the initial model presently being studied, only the dominant eigenfunction will be used in the inhomogeneous direction. This corresponds to setting $l = n = p = q = r = 1$ in equation (3.5). This is justified because the dominant eigenvalue is significantly larger than the next smallest eigenvalue and in fact, typically contains 50 percent or more of the energy. All of the higher modes exhibit this same dominance of the first eigenvalue (v. Glauser and George (1987)). The specific streamwise wavenumbers and azimuthal modes used in the initial study are shown in figure 2. For $k_1 = 0$, six azimuthal mode numbers are used and for $m = 0$ six values of $k_1 = 0$ are utilized. In addition to these, 6 other combinations are also used as can be seen from figure 2. These were selected to try and capture the maximum energy path shown in figure 1. This particular truncation results in a set of 18 complex or 36 real, ordinary differential equations. The actual equations can be found in Zheng (1990) and are not included here do lack of space.

As was discussed earlier, figure 1 indicates that there is an exchange of energy between streamwise wavenumbers and azimuthal mode numbers, and in particular, there is a maximum energy path between the two peaks. This is consistent with the mechanism for turbulence production suggested by Glauser and George (1987) and it is anticipated that the dynamical systems model will show this as well. This proposed mechanism is based on extensive azimuthal correlations taken in the near field jet shear layer at $x/D = 3$ and their subsequent breakdown into azimuthal modes. The proposed mechanism consists of 4 stages which are shown in figure 3. Briefly, vortex ring-like concentrations arise from an instability of the base flow, the induced velocities from vortices which have already formed providing the perturbation for those which follow. These pairs of rings then behave like the textbook examples or inviscid rings, the rear vortex ring overtaking the vortex ring ahead of it, the rearward vortex being reduced in radius and the forward being expanded by their mutual interaction. The rearward ring is stabilized by the reduction in its vorticity (by compression) thus the predominance of the 0th mode (seen from the correlations, v. Glauser and George (1987)) on the high speed side (core region). The forward ring has its vorticity increased by stretching as it expands in radius. This narrowing of its core while the radius is expanding causes the leading vortex to become unstable (as in the Widnall-Sullivan mechanism (v. Widnall and Sullivan (1973) and Yamada and Matsui(1978)), thus the predominance of the 4-6 modes (seen from the azimuthal correlations of Glauser and George (1987)) from the center of the shear layer outwards. The continued effect of the rearward vortex on the forward one, and the now highly distorted ring interaction with itself, accelerates the instability until its vorticity is now entirely in small scale motions, in effect an energy cascade from modes 4-6 all the way to dissipative scales. This incoherent turbulence is swept from the outside where it has been carried, back to the center of the mixing layer as the still-intact rearward vortex passes. It is this collecting of the debris, both small-scale vorticity and fluid material, which has been recognized as "pairing." The entire process is repeated as a new rearward vortex overtakes and destabilizes the one ahead of it.

CONCLUSIONS

The dynamical equations have been derived for the random coefficients in the axisymmetric jet shear layer. These equations exhibit, as expected the same type of nonlinearities (quadratic and cubic) as those of Aubry et al. (1988). It is anticipated that the dynamical systems model will show that there is a transfer of energy between the peak and surrounding streamwise wavenumbers and azimuthal modes 4,5 and 6. If this is seen to be the case it would indicate that the mechanism for turbulence production suggested by Glauser and George (1987) may be a basic mechanism for turbulence production in this flow.

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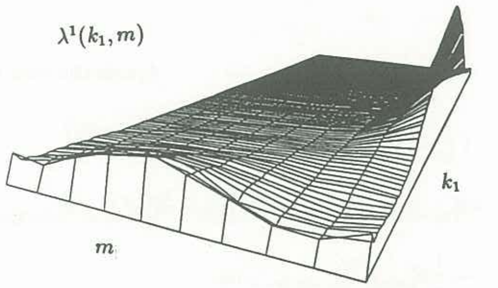

Figure 1 Dominant eigenvalue $\lambda^1(k_1, m)$ plotted as a function of streamwise wavenumber k_1 and azimuthal mode number m.

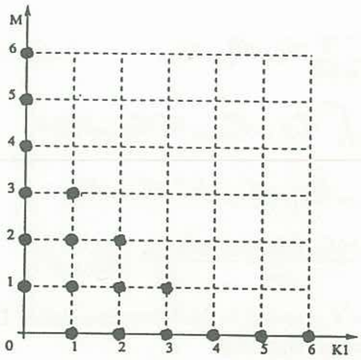

Figure 2 Modes retained in the dynamical systems model.

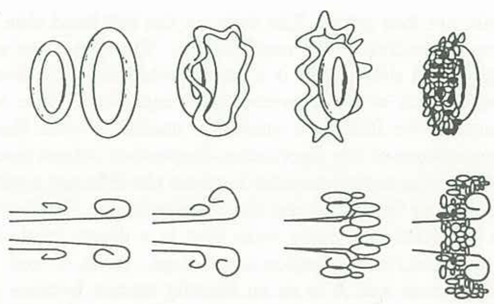

Figure 3 The proposed 4 stages of turbulence production.