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ON THE ROLE OF MEAN-FLOW THREE-DIMENSIONALITY IN TURBULENCE MODELLING

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ABSTRACT

A new modelling concept is developed as a tool to take into account the role of mean-flow three-dimensionality in turbulence modelling. This concept is based on a thermodynamic analogy. Application of the concept to the turbulent boundary layer on an infinite swept wing leads to identifying two effects of mean-flow three-dimensionality that cause the normalized magnitude of the shear-stress vector to deviate from its value at the start of the pressure gradient. Comparison of calculated and experimental results for the local skin-friction coefficient gives satisfactory agreement.

INTRODUCTION

A broad conclusion at the international symposium on "Perspectives in Turbulence Studies", held at Göttingen 1987, was (Cousteix et al. (1987)): Existing turbulence models including the stress equation models have been developed on a basis of experimental data concerning two-dimensional ('2D') flows; effects of mean-flow three-dimensionality are not well taken into account.

The present paper is a contribution to overcome this drawback.

THERMODYNAMIC ANALOGY

Consider a moving fluid element in a turbulent flow of a Newtonian fluid of constant density ρ and constant kinematic viscosity ν, in the absence of mass diffusion and chemical reactions. The entropy source strength in the balance equation for the entropy of the moving fluid element can be written as a sum of products, each product having the form independent thermodynamic flux times independent thermodynamic force. In thermodynamics of irreversible processes (Meixner et al. (1959)) the independent fluxes are postulated to be functions of the independent forces. Expanding these functions by Taylor's series about independent forces equal to zero and truncating after the linear terms, gives so-called (linear) phenomenological equations. These equations comprise the material laws of the above-mentioned fluid.

Consider now an element of volume moving with the field of the ensemble-averaged fluid velocity. We exclude turbulent fluctuations of an external force per unit mass appearing in an inertial frame of reference. Using Cartesian tensor notation the turbulent kinetic-energy equation for the moving element of volume reads

$$\frac{\partial e}{\partial t} + U_j \frac{\partial e}{\partial x_j} = -\overline{u_j u_i}\frac{\partial U_i}{\partial x_j} - \frac{\partial J_j/\rho}{\partial x_j} - \varepsilon. \qquad (1)$$

Here t, x_j, U_j, e, J_j, ε are, respectively, time, Cartesian coordinate, x_j-component of ensemble-averaged fluid velocity, turbulent kinetic energy, x_j-component of turbulent-energy flux, dissipation of turbulent energy. The overbar denotes ensemble averaging, and $u_i = \tilde{u}_i - U_i$ where $\tilde{u}_i$ is the x_i-component of the instantaneous fluid velocity. Introduce an extensive quantity $\int \rho s dV$ where dV is an element of volume and $ds = de/T$. The denominator T is an unspecified (scalar-valued) function of e only for a given ρ and ν. A relation for the change of the scalar quantitiy s in the moving element of volume is

$$\frac{\partial s}{\partial t} + U_j \frac{\partial s}{\partial x_j} = \frac{1}{T}\left(\frac{\partial e}{\partial t} + U_j \frac{\partial e}{\partial x_j}\right). \qquad (2)$$

Introducing equation ('eq.') (1) into eq.(2) yields, after some manipulations, the balance equation

$$\rho\left(\frac{\partial s}{\partial t} + U_j \frac{\partial s}{\partial x_j}\right) = -\frac{\partial}{\partial x_j}\left(\frac{J_j}{T}\right) + \sigma, \qquad (3)$$

$$\sigma = J_j \frac{\partial(1/T)}{\partial x_j} + \left[\frac{\rho}{T}(-\overline{u_i u_j}) + \frac{\rho}{T}\frac{2e}{3}\delta_{ij}\right] \times \left[\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right] + (-\rho\varepsilon)\frac{1}{T} \qquad (4)$$

for the extensive quantity $\int \rho s dV$ in the moving element of volume, δ_{ij} is the Kronecker delta. The source strength σ can be written in terms of independent "turbulent fluxes and forces":

$$\sigma = \sum_{\lambda=1}^{m} Y_\lambda X_\lambda. \qquad (5)$$

The m independent fluxes Y_λ are the 3 components J_j, 5 independent components $(\rho/T)(-\overline{u_i u_j}) + (\rho 2e\delta_{ij})/(3T)$ and the scalar flux $\rho(-\varepsilon)$. The independent forces X_λ arise in eq.(4), after some manipulations, as coefficients of the independent fluxes. In the moving element of volume, the independent fluxes at time t, $Y_\lambda(t)$, are postulated to be functions solely of the independent forces at a time $t^* \neq t$, $X_\lambda(t^*)$, and of parameters unspecified so far:

$$Y_\lambda(t) = Y_\lambda(X_1(t^*), \ldots, X_m(t^*)), \quad \lambda = 1, \ldots, m \qquad (6)$$

with

$$0 = Y_\lambda(0, \ldots, 0), \quad \lambda = 1, \ldots, m. \qquad (7)$$

Here the parameters are omitted. Note that the mechanism that links independent turbulent fluxes to independent turbulent forces is not of local nature in contrast to the mechanism that links independent thermodynamic fluxes to independent thermodynamic forces. The functions Y_λ are mathematical forms composed of mathematical operations (free of dimensions) and fixed numbers only. Expanding the right-hand side ('RS') of eq.(6) about $X_j(t^*) = 0$, $j = 1, \ldots m$ by Taylor's series and truncating after the linear terms, gives:

$$Y_i(t) = \sum_{j=1}^{m} L_{ij} X_j(t^*), \quad i = 1, \ldots m. \qquad (8)$$

The coefficients L_{ij} depend solely on the above-mentioned parameters. An equivalent version of the eqs.(8) is (in Cartesian tensor notation):

$$\rho(-\varepsilon) = L^{(1)}T^{-1} + L_i^{(1)} \frac{\partial T^{-1}}{\partial x_i} + L_{ij}^{(1)} D_{ij}, \qquad (9)$$

$$J_i = L_i^{(2)}T^{-1} + L_{ij}^{(2)} \frac{\partial T^{-1}}{\partial x_j} + L_{ijk}^{(2)} D_{jk}, \quad (i = 1,2,3) \qquad (10)$$

$$\frac{\rho}{T}\left(\frac{2e\delta_{ij}}{3} - \overline{u_i u_j}\right) = L_{ij}^{(3)}T^{-1} + L_{ijk}^{(3)} \frac{\partial T^{-1}}{\partial x_k} + L_{ijkl}^{(3)} D_{lk}, \quad (i,j = 1,2,3) \qquad (11)$$

where D_{ij} is a Cartesian component of the mean strain rate **D**. Table 1 informs about the tensorial character of the coefficients in eqs.(9) to (11), which satisfy the compatibility relations (j,k,l,n = 1,2,3)

$$\begin{aligned} &L_{jk}^{(1)} = L_{kj}^{(1)}, \ L_{jk}^{(3)} = L_{kj}^{(3)}, \ L_{jlk}^{(2)} = L_{jkl}^{(2)}, \\ &L_{jkl}^{(3)} = L_{kjl}^{(3)}, \ L_{jkln}^{(3)} = L_{jknl}^{(3)}, \ L_{ii}^{(1)} = 0, \\ &L_{jkln}^{(3)} = L_{kjln}^{(3)}, \ L_{jii}^{(2)} = L_{ii}^{(3)} = L_{iij}^{(3)} = 0, \\ &L_{iijk}^{(3)} = L_{jkii}^{(3)} = 0. \end{aligned} \qquad (12)$$

Note that, in eqs.(9) to (11), the fluxes are to be taken in the moving element of volume at time t, whereas the forces T^{-1}, $\partial T^{-1}/\partial x_i$, D_{ij} are to be taken in the moving element of volume at time t*. The coefficients in eqs.(9) to (11) are functions solely of the above-mentioned parameters, where the term "function" is to be understood in the same sense as with Y_λ. Assume now that the system (9) to (11) is isotropic. Invariance under inversion requires (i,j,k = 1,2,3):

$$L_i^{(1)} = L_i^{(2)} = L_{ijk}^{(2)} = L_{ijk}^{(3)} = 0. \qquad (13)$$

A further consequence of the isotropy of the system (9) to (11) is (cf. Batchelor (1953))

$$\begin{aligned} &L_{ij}^{(1)} = L_{ij}^{(3)} = 0, \quad L_{ij}^{(2)} = L^{(2)}\delta_{ij}, \\ &L_{ijkl}^{(3)} = I^{(3)}\delta_{ij}\delta_{kl} + J^{(3)}\delta_{ik}\delta_{jl} + K^{(3)}\delta_{il}\delta_{jk} \end{aligned} \qquad (14)$$

for i,j,k,l = 1,2,3. Note that the tensors $L_{ij}^{(1)}$ and $L_{ij}^{(3)}$ have a zero trace. The coefficients $L^{(2)}$, $I^{(3)}$, $J^{(3)}$, $K^{(3)}$ are scalars. Introducing eqs.(13), (14) into eqs.(9) to (11) yields (i,j = 1,2,3):

$$\rho(-\varepsilon) = L^{(1)}T^{-1}, \quad J_i = L^{(2)} \frac{\partial T^{-1}}{\partial x_i}, \qquad (15)$$

$$-\overline{u_i u_j} + \frac{2}{3} e\delta_{ij} = L D_{ij}, \quad D_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \qquad (16)$$

$$\text{where} \quad L = (T/\rho)\left(J^{(3)} + K^{(3)}\right) \qquad (17)$$

is a scalar.

Tensorial order	0	1	2	3	4
Cartesian	$L^{(1)}$	$L_i^{(1)}$	$L_{ij}^{(1)}$	$L_{ijk}^{(2)}$	$L_{ijkl}^{(3)}$
components of		$L_i^{(2)}$	$L_{ij}^{(2)}$	$L_{ijk}^{(3)}$	
polar tensors			$L_{ij}^{(3)}$		

Table 1. Tensor character of coefficients.

MODELLING CONCEPT AND UNIVERSAL FUNCTIONS

The system of eqs.(15) to (17) is adopted. The - so far unspecified - parameters that enter the coefficients $L^{(1)}$, $L^{(2)}$, L are solely ρ, $\rho\nu$, e, the second invariant of the mean strain rate, i_2, and an appropriate (frame-indifferent, scalar) length R; the three variable parameters are to be taken in the moving element of volume at time t.

A formal representation of the coefficients in eqs.(15), (16) is (n = 1,2,3):

$$L_n = \Phi_n(\rho, \rho\nu, e, i_2, R), \qquad (18)$$

where $L_1 = L^{(1)}$, $L_2 = L^{(2)}$, $L_3 = L$, and the Φ_n - like T - are functions in the same sense as Y_λ; each one is universal, i.e., it is the same for all mean flows within the scope of present paper, it is independent of the frame of reference and is left unaltered by rotation and inversion of the Cartesian coordinate system associated with the frame of reference. Dimensional reasoning yields that the RS of eq.(18) for n = 3 - in the International System of Measurement, say - cannot depend on ρ and $\rho\nu$ separately but on the combination $(\rho\nu)/\rho$ only. Hence, eq.(18) for n = 3 can be written

$$L = \Phi(\nu, e, -i_2, R), \qquad (19)$$

where Φ is a function in the same sense as Y_λ and is universal. With the aid of the Π-theorem (cf. Romberg (1985)) the formula (19) can be reduced to relationships between dimensionless combinations:

$$L = \sqrt{(-i_2)}\, R^2 \Phi(\Pi_1, \Pi_2, 1, 1) \ \text{if} \ \infty > -i_2, R > 0, \qquad (20)$$

$$\text{where } \Pi_1 = \nu/\left(\sqrt{(-i_2)}\, R^2\right), \quad \Pi_2 = e/(-i_2 R^2).$$

$$L = \sqrt{e}\, R \Phi\left(\Pi_1 \Pi_2^{-1/2}, 1, \Pi_2^{-1}, 1\right) \quad \text{if} \ \infty > e, R > 0. \qquad (21)$$

Equating the RS of eqs.(20) and (21) leads to

$$\Phi(\Pi_1, \Pi_2, 1, 1) = \Pi_2^{1/2} \Phi\left(\Pi_1 \Pi_2^{-1/2}, 1, \Pi_2^{-1}, 1\right) \quad \text{if} \ \infty > e, -i_2, R > 0. \qquad (22)$$

Π_2 can be eliminated from eq.(20) with the aid of eq.(22), one obtains

$$L = \sqrt{(-i_2)}\, R^2 F(\Pi_1) \quad \text{if} \ \infty > e, -i_2, R > 0, \qquad (23)$$

where F is a function in the same sense as Y_λ and is universal.

TURBULENT BOUNDARY LAYER ('BL') ON AN INFINITE SWEPT WING

Consider a statistically steady mean flow over a swept-back wing at zero incidence in the absence of external forces. The wing is a (smooth) flat plate of infinite span whose surface is generated by parallel straight lines of α sweep, $\alpha \leq 45°$, say. The x_3-axis coincides with the leading edge of the swept wing. The positive x_1-axis lies in the wing surface, x_2 is the normal distance from the wing surface. The frame of reference is taken to be the wing itself. It is assumed: The BL over a front portion $0 \leq x_1 \leq x_{1s}$ of a wing side - in a zero pressure gradient - is 2D in planes perpendicular to wing surface and parallel to the direction of the oncoming stream, and fully developed at $x_1 = x_{1s}$. The BL over the front portion is followed downstream by a turbulent BL in a non-zero pressure gradient in positive x_1-direction, in which infinite-swept-wing conditions prevail. The pressure gradient over the wing side is continuous.

The mathematical problem associated with the quantitative description of the (attached) 3D BL on the wing side includes the design of an appropriate turbulence model. A generally accepted qualitative picture of the 3D BL includes (cf. Cousteix et al. (1987)): In a moving element of volume the direction of the shear-stress vector $(-\rho\overline{u_1u_2}, \ -\rho\overline{u_3u_2})$ lags behind the direction of the velocity-gradient vector $(\partial U_1/\partial x_2, \ \partial U_3/\partial x_2)$.

Application of the modelling concept to the mean-flow over the wing side leads to eq.(16) where the left-hand side and the coefficient L are to be taken at time t whereas D_{ij} is to be taken at time t*. Let $x_1(t)$ and $x_1(t^*)$ be, respectively, the x_1-coordinate of the location of the moving element of volume at time t and t*. The difference

$$d = x_1(t) - x_1(t^*) \tag{24}$$

depends on $x_1 = x_1(t)$, but not on t, on the streamline that the moving element of volume is following. Note that the mean flow over the wing is statistically steady. Eq.(24) is to be interpreted as a representation of a frame-indifferent scalar d in a special frame of reference associated with a special Cartesian coordinate system.

To the boundary-layer approximation, eq.(16) can be replaced by

$$-\overline{u_i u_2} = \frac{L}{2}\frac{\partial U_i}{\partial x_2}, \quad i = 1,3 \tag{25}$$

where $t^* \leq t$, $d \geq 0$, according to above qualitative picture. The mathematical form (25) with $L > 0$ is able to reproduce the lag included in the above qualitative picture.

An expression for L appropriate to the 2D fully developed turbulent BL upstream of the start of the pressure gradient is due to Cousteix et al. (1972). With $-4i_2 = (\partial U_1/\partial x_2)^2 + (\partial U_3/\partial x_2)^2$ and the assumption R equal to the mixing length, this expression can be written

$$\frac{L}{4R^2\sqrt{-i_2}} = \tag{26}$$

$$\left[1 - \exp\left\{-\frac{1}{10.66}\left[2\Pi_1^{-1} + \frac{L}{R^2\sqrt{-i_2}}\Pi_1^{-2}\right]^{\frac{1}{2}}\right\}\right]^2,$$

and eq.(26) should then be a - at least approximate - representation of the unknown universal function $F(\Pi_1)$. We are left with the problem to develop appropriate expressions for d and R in the (attached) turbulent 3D BL.

Expression for d

On a streamline within the 3D BL, $\mathbf{D}(x_1)$ (after the BL-solution) is assumed neither singular nor to exhibit rotational symmetry which would render the direction of two eigenvectors $\mathbf{e}_l(x_1)$, $l = 1,2,3$ - with the Cartesian components $e_{li}(x_1)$ - of $\mathbf{D}(x_1)$ indefinite, we then have for the eigenvalue $\lambda_l(x_1)$ associated with the direction of $\mathbf{e}_l(x_1)$, $l = 1,2,3$:

$$\frac{\lambda_2(x_1)}{\lambda_1(x_1)} + \frac{1}{2} = -\frac{\lambda_3(x_1)}{\lambda_1(x_1)} - \frac{1}{2} = A(x_1) = \frac{i\sqrt{3}}{2}\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}, \tag{27}$$

$$\text{where} \qquad \lambda_1(x_1) = \alpha_1 + \alpha_2, \tag{28}$$

$$\alpha_1 = \left[\frac{i_3(x_1)}{2} + \left\{\left(\frac{i_2(x_1)}{3}\right)^3 + \left(\frac{i_3(x_1)}{2}\right)^2\right\}^{\frac{1}{2}}\right]^{\frac{1}{3}}, \tag{29}$$

$$\alpha_2 = \left[\frac{i_3(x_1)}{2} - \left\{\left(\frac{i_2(x_1)}{3}\right)^3 + \left(\frac{i_3(x_1)}{2}\right)^2\right\}^{\frac{1}{2}}\right]^{\frac{1}{3}}, \tag{30}$$

i is the imaginary unit and $i_3(x_1) = \det \mathbf{D}(x_1)$ is the third invariant of $\mathbf{D}(x_1)$.

If, in the 3D mean-flow over the wing, $A(x_1)$ and the direction of $\mathbf{e}_l(x_1)$, $l = 1,2,3$ were constant on a section of a streamline then, on that section, the ratio of any two $D_{ij}(x_1)$ (at the same location) were constant and - the section being sufficiently long - $d(x_1)$ would eventually become zero in downstream direction as a result of a process of adjustment between the Reynolds stress and the mean strain rate.

In an element of volume moving with the BL mean-velocity-field along the streamline within the 3D BL, it is the rate of change of A, $\dot{A}$, and the rate of change of ratios e_{li}/e_{lj}, $i \neq j$, $\dot{E}_{lij}$, that affect the evolution of d. In mathematical terms this suggests: The frame-indifferent scalar $d(\Delta)$ - where $\Delta = x_1 - x_{1s}$ can be interpreted as a representation of a frame-indifferent scalar Δ - a distance - in a special frame of reference associated with a special Cartesian coordinate system - is a scalar-valued isotropic functional $\mathbf{F}$ - independent of the frame of reference - in the history of the frame-indifferent scalar $\dot{A}$ in the element of volume while moving along the streamline section $0 < \vartheta \leq \Delta$, i.e.,

$$d(\Delta) = \mathbf{F}\left(\sqrt{Re}\,\dot{A}(\vartheta); \Delta\right), \quad 0 < \vartheta \leq \Delta, \tag{31}$$

$$0 = \mathbf{F}\left(\sqrt{Re}\,\dot{A}(\vartheta); \Delta\right) \text{ for } \dot{A}(\vartheta) = 0, \ 0 < \vartheta \leq \Delta \text{ or } \Delta = 0 + 0,$$
$$\mathbf{F}\left(\sqrt{Re}\,\dot{A}(\vartheta); \Delta\right) \rightarrow 0 \text{ as } x_2 \rightarrow \gamma \ \text{ if } \dot{A}(\vartheta) \rightarrow 0 \text{ as } x_2 \rightarrow \gamma \tag{32}$$

$$\text{with } \vartheta \text{ fixed}, \ 0 < \vartheta \leq \Delta, \ \gamma = 0 \text{ or } \infty. \tag{33}$$

Here ϑ is a dummy variable for Δ. Mathematical reasoning suggests that the supposed properties of $\mathbf{F}$ are incompatible with letting $\mathbf{F}$ depend on quantities $\dot{E}_{lij}(\vartheta)$, $i \neq j$. Further properties to be imposed on $\mathbf{F}$ are: 1.Reflexion of the 3D BL flow in the wing plane leaves $\mathbf{F}$ unaltered. 2.Perturbation theory suggests: The limit of $d(\Delta)$ as

$$Re \rightarrow \infty \ \text{ with } x_1/\Lambda, \ (x_2\sqrt{Re})/\Lambda, \ \mathbf{u}_\infty, \ \Lambda \text{ fixed} \tag{34}$$

is 0[1] (i.e. different from zero and finite). Here Λ, $\mathbf{u}_\infty$ are, respectively, the finite wing-chord length, external-stream velocity in zero pressure gradient. $Re = (|\mathbf{u}_\infty|\Lambda)/\nu$ can be interpreted as a special representation of a frame-indifferent scalar Re (cf. Romberg (1985a)). - It is seen from the cubic eq. for $\lambda_l(x_1)$, $l = 1,2,3$, and

$$i_2(x_1)/Re, \ i_3(x_1)/Re = 0[1]$$

in the limit process (34), that

$$\lambda_m(x_1)/\sqrt{-i_2(x_1)} \rightarrow 1, \ \lambda_n(x_1)/\sqrt{-i_2(x_1)} \rightarrow -1,$$
$$(\lambda_p(x_1)i_2(x_1))/i_3(x_1) \rightarrow 1$$

in the limit process (34). We have to specify $m = 1$, $n = 2$, $p = 3$ in line with the formulation (27) to (30). Hence

$$\sqrt{Re}\,\dot{A}(\vartheta) = 0[1]$$

in the limit process (34).

A tentative suggestion for $\mathbf{F}$ is

$$\mathbf{F}\left(\sqrt{Re}\,\dot{A}(\vartheta); \Delta\right) = \int_0^\Delta \sqrt{Re}\,\dot{A}(\vartheta)G(\Delta,\vartheta)d\vartheta, \ \Delta > 0, \tag{35}$$

where the function $G(\Delta,\vartheta)$ is scalar-valued, isotropic and independent of the frame of reference.

Expression for R

An appropriate expression for the 2D fully developed turbulent BL is (cf. Cousteix et al. (1987))

$$R = C\delta \tanh[(\kappa x_2)/(C\delta)] \tag{36}$$

$$\text{where} \quad C = C_0 = 0.085, \quad \kappa = \kappa_0 = 0.41 \qquad (37)$$

are used for the coefficients C, κ. The BL thickness can be redefined to become a frame-indifferent scalar δ (cf. Romberg (1985a)). x_1 can be interpreted as a special representation of a frame-indifferent scalar X. Denote by X' a dummy variable for X. On a streamline within the 3D turbulent BL it is the frame-indifferent departure $\mathbf{u}_\infty - \mathbf{u}_e(X')$ $0 < X' \leq X$ - where $\mathbf{u}_e$ is the BL-edge velocity - that affects $C(X) - C_0$ and $\kappa(X) - \kappa_0$. In mathematical terms this suggests:.

$$\kappa(X) - \kappa_0 = \psi\,(\mathbf{u}_\infty - \mathbf{u}_e(X'); X), \quad 0 \leq X' \leq X, \quad (38)$$

$$C(X) - C_0 = \varphi\,(\mathbf{u}_\infty - \mathbf{u}_e(X'); X), \quad 0 \leq X' \leq X, \quad (39)$$

where ψ and φ are scalar-valued, isotropic functionals independent of the frame of reference. If the RS of eqs.(38), (39) does not exhibit an appreciable memory of the departure $\mathbf{u}_\infty - \mathbf{u}_e(X')$ upstream of X, and if - for a fixed sweep angle - $\mathbf{u}_\infty - \mathbf{u}_e(X)$, X, and ν can be considered the only physical quantities that enter the RS of eqs.(38), (39) we can conclude, utilizing a representation theorem for scalar-valued, isotropic functions of one polar vector (cf. Müller (1973)) and dimensional analysis:

$$\kappa(X) - \kappa_0 = \psi(\Pi), \; C(X) - C_0 = \varphi(\Pi), \\ \Pi = \sqrt{[\mathbf{u}_\infty - \mathbf{u}_e(X)]^2}\, X \nu^{-1} \qquad (40)$$

where ψ and φ are scalar-valued functions independent of the frame of reference.

TESTS AND CONCLUSIONS

The measurements shown in figure 1 were made in the turbulent BL on a 35° swept, infinite wing simulated in a rig at the Netherlands NLR (cf. Schneider (1977)).

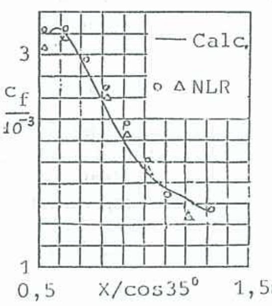

Figure 1 Comparison of calculated and experimental skin-friction coefficients c_f (based on $0.5\rho\mathbf{u}_e^2$).

This BL reached separation at X = 1.31×(cos 35°)m. The calculation of the full line is based on the linear approximation of eqs.(40) about $\Pi = 0$ where $\kappa_0^{-1}(d\psi/d\Pi)_{\Pi=0} = -0.277 \cdot 10^{-6}$, $C_0^{-1}(d\varphi/d\Pi)_{\Pi=0} = -5.93 \cdot 10^{-7}$, on eq.(36), on eq.(26) where $-4i_2 = [\partial U_1/\partial x_2]^2 + [\partial U_3/\partial x_2]^2$, and on eqs.(25) for zero "displacement" d. It is the result

$$\dot{A}(\vartheta), \partial\dot{A}(\vartheta)/\partial x_2 \to 0 \\ d(\Delta) \to 0 \text{ as } x_2 \to 0 \\ \text{with } \vartheta \text{ fixed}, 0 < \vartheta \leq \Delta \qquad (41)$$

that encouraged to do this test. The discrepancy between experimental and theoretical results shown in figure 1 may be due - at least partially - to the approximation to infinite-swept conditions of the NLR rig. This approximation seems to be rather worse than that of the Bradshaw Pontikos rig (cf. Bradshaw et al. (1985)). - It is worth noting that the present theory also predicts

$$d(\Delta) \to 0 \text{ as } x_2 \to \infty \text{ with } \Delta \text{ fixed} \qquad (42)$$

if $i_3(x_1) \to 0$, $i_2(x_1) \to \Gamma(x_1) \neq 0$ as $x_2 \to \infty$ with ϑ fixed, $0 < \vartheta \leq \Delta$ where $\Gamma(x_1)$ is a limiting value. Apparently, reliable experimental results for testing the predictions (41), (42) are not available (cf. Bradshaw et al. (1985)).

Define a normalized shear-stress magnitude, τ':

$$\tau'(x_1, x_2) = \tau(x_1, x_2)/[\rho K(x_1, x_2) g^2(x_1, x_2) R_0^2(x_1, x_2)]. \quad (43)$$

Here g, τ, R_0, K are, respectively, the magnitude of the velocity-gradient and of the shear-stress vector, R after formulae (36), (37), and the RS of eq.(26) where $-4i_2 = g^2$. With the aid of eqs.(25), (26) we obtain

$$\tau' = [R(x_1, x_2)/R_0(x_1, x_2)]^2 [g(x_1 - d(x_1, x_2), x_2)/g(x_1, x_2)]. \quad (44)$$

Here R means R after the formulae used in the calculation of the full line in Figure 1. - Consider now the Bradshaw Pontikos flow (1985) between the first and the last measurement station on a line parallel to the tunnel axis. Two effects of mean-flow three-dimensionality are identified that cause τ' to deviate from 1. First, the departure of the BL-edge mean-velocity in a positive chordwise pressure gradient from the corresponding velocity in a zero pressure gradient reduces τ'. Note that $[R(x_1, x_2)/R_0(x_1, x_2)]^2$ in eq.(44), with $x_2/\delta(x_1) > 0$ fixed, decreases as $x_1 \geq x_{1s}$ increases; $[R(x_1, x_2)/R_0(x_1, x_2)]^2$ in eq.(44) is equal to 0.507 for $x_2/\delta(x_1) = 0.6$ at the last measurement station. Second, the evolution of the displacement $d(x_1, x_2)$ in an element of volume, following a streamline in a positive chordwise pressure gradient, causes the second factor on the RS of eq.(44) to deviate from one in this element of volume. A simple calculation, that relies on mean-velocity-magnitude profiles and on results for the angle between local mean-velocity and velocity-gradient vector reported by Bradshaw et al. (1985), predicts that $g(x_1, x_2)$ decreases, as x_1 increases, downstream of the first measurement station along a streamline with $x_2/\delta(x_1) = 0.5$ at this station. Hence, at a given location on this streamline the second effect goes opposite to the first one.

Additional, quantitatively noteworthy experimental results - e.g. for the shear-stress vector - are needed in further developing the present theory. A key-aspect will be: The present modelling concept should be understood as an approximation to a more general modelling concept that relies on a rather more general eq.(6)

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