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# EFFECTS OF ROTATION ON HOMOGENEOUS TURBULENCE

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**Abstract**

Turbulence models are known to have difficulty in flows with strong rotation. In this paper we examine the effect of rotation, using rapid distortion theory (RDT) for homogeneous turbulence as a guide. It is shown that rotation significantly modifies the turbulent stress anisotropy in a way not predicted by current turbulence models. The reasons for this failure of models is argued to be a lack of information about the turbulence structure. A new tensor quantity, the *structure tensor*, is defined and shown to be of critical importance in the rotation problem. The physical reasons for the effects of rotation are explained, and a simple model that does display the effects is proposed.

## 1. Introduction

Rotation is known to reduce the dissipation rate, an effect only recently incorporated in some turbulence models (Bardina *et al.* 1985). It had been thought (*eg.* Speziale 1981) that rapid rotation would cause a Taylor-Proudman reorganization of turbulence into a two-dimensional (2D) state, but a direct numerical simulation of isotropic turbulence refuted this idea (Speziale *et al.* 1987).

A subsequent numerical simulation with an initially anisotropic field, by Mansour, produced the shocking result that the anisotropy of the turbulent stresses was rapidly reduced by rotation. This calculation, an example of which is shown in Fig. 1, provided the primary impetus for the present work. Similar effects have been obtained using EDQNM (Cambon and Jacquin 1989).

Since the simulations are of necessity at low Reynolds numbers, there was a suggestion that this might simply be a low Reynolds number effect. To check this possibility, an inviscid RDT was carried out by the author with the help of T.S. Shih, and this work showed that the observed effect was due to inertial and not viscous effects. As we shall see, this effect is *not* mirrored by current turbulence models.

This led the author to think long and hard about how turbulence models models can be modified to incorporate the effects of rotation, and ultimately to develop some new ideas for turbulence modeling. The primary objective of this paper is to report these new ideas.

## 2. Anisotropy modeling in homogeneous turbulence

Contemporary one-point turbulence models are based on the transport equations for the turbulent stresses $R_{ij} = \overline{u_i' u_j'}$. The state of the turbulence is then characterized by the turbulent kinetic energy $q^2/2$, where $q^2 = R_{ii}$, and the turbulent stress anisotropy tensor

$$b_{ij} = \frac{R_{ij} - q^2\delta_{ij}/3}{q^2}. \qquad (2.1)$$

For incompressible homogeneous turbulence the evolution equation for $b_{ij}$ can be written as

$$\dot{b}_{ij} = -\frac{2}{3}S_{ij} - (b_{ik}S_{kj} + b_{jk}S_{ki} - \frac{2}{3}b_{nm}S_{nm}\delta_{ij}) + 2b_{nm}S_{nm}b_{ij}$$
$$+ (b_{ik}\Omega_{kj} + b_{jk}\Omega_{ki})$$
$$+ \frac{1}{q^2}[T_{ij} - (D_{ij} - \frac{1}{3}D_{kk}\delta_{ij})] + 2\frac{\varepsilon}{q^2}b_{ij}. \qquad (2.2)$$

Here $S_{ij} = (U_{i,j} + U_{j,i})/2$ is the mean strain-rate, $\Omega_{ij} = (U_{i,j} - U_{j,i})/2$ is the mean rotation-rate, $T_{ij}$ the pressure-strain term, $D_{ij}$ the viscous dissipation term, and $\varepsilon$ is the kinetic energy dissipation rate. Subscripts below commas denote partial differentiation with respect to that coordinate, and an overdot denotes $\partial/\partial t$.

The pressure-strain term is split into a rapid part involving the mean velocity gradients and a slow term involving non-linear turbulence interactions. The rapid pressure-strain term may be expressed directly as

$$T_{iq}^{(r)} = 2U_{p,j}(M_{ijpq} + M_{qjpi}) \qquad (2.3)$$

where

$$M_{ijpq} = \int \frac{k_p k_q}{k^2} E_{ij}(\mathbf{k}) d^3\mathbf{k}. \qquad (2.4)$$

Here $E_{ij}(\mathbf{k})$ is the turbulent velocity spectrum tensor. A great deal of effort (including the author's) has gone in to developing elaborate models of $\mathbf{M}$ that meet various conditions it is known to satisfy. In every case $\mathbf{M}$ has been modeled in terms of $\mathbf{b}$ and $q^2$, and invariably one has to adjust remaining free coefficients to match experiments.

One purpose of this paper is to point out that these models fail in one of the simplest building-block flows, namely homogeneous turbulence subjected to mean rotation without strain. In this case the most general model of this type produces an evolution equation of the form

$$\dot{b}_{ij} = C_1(\Omega_{ki}b_{kj} + \Omega_{kj}b_{ki}) + C_2(\Omega_{ki}b_{kj}^2 + \Omega_{kj}b_{ki}^2)$$
$$+ C_3\Omega_{pq}(b_{iq}b_{pj}^2 + b_{jq}b_{pi}^2) + \ldots. \qquad (2.5)$$

Here the dots indicate dissipation and slow pressure-strain terms that do not depend explicitly on the mean rotation and which are not important for sufficiently strong rotation rates. The coefficients are functions of the invariants of $\mathbf{b}$, namely $\mathrm{II} = -b_{ij}b_{ji}/2$ and $\mathrm{III} = b_{ij}b_{jk}b_{ki}/3$. Note that, since $\Omega_{ij}$ is antisymmetric and all tensors formed from $b_{ij}$ are symmetric, the mean rotation rate drops out of the equations for these invariants. Hence, models of this type predict no effect of rapid rotation on these invariants. As we shall see, this is not correct; *hence there is a fundamental flaw in the turbulence models.*

The basic problem is that b contains information about the *componentality* of the turbulence, but does not contain any information about its physical structure or *dimensionality*. For example, if $b_{11} = -1/3$ then we know that $u'_1 = 0$ everywhere, which means only that the turbulence is two-component (2C). The other two components could vary in one, two, or even 3 directions, so it is not correct to conclude that the turbulence is also 2D. The problem is especially acute for 1C turbulence, for which the **M(b)** model displays a singularity arising from different possible structures having the same b (Reynolds 1989).

## 3. The structure tensor

In order to provide some additional information to resolve this problem, we introduce the *structure tensor*,

$$Y_{ij} = \overline{\Psi'_{n,i}\Psi'_{n,j}}. \tag{3.1}$$

$\Psi'_n$ is the *vector stream function* defined by

$$u'_i = \epsilon_{ijk}\Psi'_{k,j} \tag{3.2a}$$

and the auxiliary condition (Aris 1962)

$$\Psi'_{i,i} = 0. \tag{3.2b}$$

It is easily demonstrated that $\Psi'_i$ satisfies

$$\Psi'_{i,jj} = -\epsilon_{ijk}u'_{k,j} = -\omega'_i. \tag{3.3}$$

For homogeneous turbulence,

$$Y_{ij} = \int \frac{k_i k_j}{k^2} E_{nn}(\mathbf{k}) d^3\mathbf{k}. \tag{3.4}$$

Note that $Y_{ii} = q^2$. If one of the principal values $Y_{\alpha\alpha}$ is zero, then there is no energy associated with any mode for which $k_\alpha \neq 0$, and so the turbulence is independent of $x_\alpha$. Thus, whereas $R_{ij}$ carries information about the *componentality* of the turbulence, $Y_{ij}$ carries information about its *dimensionality* and eddy structure.

The anisotropy of $Y_{ij}$ is given by the *structure anisotropy tensor*

$$y_{ij} = \frac{Y_{ij} - q^2\delta_{ij}/3}{q^2}. \tag{3.5}$$

Under arbitrary irrotational strain with $S_{11} > S_{22} \geq S_{33}$ the vortex lines eventually become straight and aligned with $x_1$, and so the turbulence becomes independent of $x_1$. A value of $y_{\alpha\alpha}$ close to $-1/3$ indicates that the energy-containing eddies are relatively long in the $x_\alpha$ direction, while a value close to 2/3 would indicate that they are relatively short in the $x_\alpha$ direction.

RDT for arbitrary irrotational strain shows that $y_{ij} = b_{ij}$ at both the start and end of the deformation, which suggests that y may not be essential in modeling of rapidly strained flows. However, RDT for rotation of inhomogeneous turbulence (section 4) shows that y is unchanged and that $b_{ij}$ is driven to $-y_{ij}/2$. Therefore, y clearly should be involved in turbulence models for rotating flows.

## 4. RDT for rotation of homogeneous turbulence

### 4.1 Evolution equations

We consider incompressible turbulence subjected to solid body mean rotation given by

$$U_1 = \Gamma x_2 \qquad U_2 = -\Gamma x_1. \tag{4.1}$$

The analysis is most easily done in a frame rotating with the mean motion. Denoting the coordinates of this frame by $\xi_i$, time in this frame by $\tau$, and the fluctuation velocities in this frame by $v_i$, the momentum equations under the inviscid RDT approximations become

$$\frac{\partial v_i}{\partial \tau} = -\frac{1}{\rho}\frac{\partial p}{\partial \xi_i} - 2\epsilon_{ij3}v_j\Gamma \tag{4.2}$$

and the continuity equation is

$$\frac{\partial v_i}{\partial \xi_i} = 0. \tag{4.3}$$

The velocity field can be represented by a Fourier series in a large box surrounded by replicates,

$$v_i(\underline{\xi}, \tau) = \sum_{\underline{\kappa}} \hat{v}_i(\underline{\kappa}, \tau)e^{-i\kappa_n\xi_n} \tag{4.4}$$

where $\underline{\kappa}$ is the wavenumber in the rotating frame, and the sum is over all wavenumbers that fit the box. A similar representation is used for $p$. The equations for the Fourier amplitudes are then

$$\frac{d\hat{v}_i}{d\tau} = \frac{i}{\rho}\kappa_i\hat{p} - 2\epsilon_{ij3}\Gamma\hat{v}_j \tag{4.5}$$

and continuity becomes

$$\kappa_i\hat{v}_i = 0. \tag{4.6}$$

Applying (4.6), one finds

$$\frac{i}{\rho}\hat{p} = \frac{2\Gamma\epsilon_{ij3}\kappa_i\hat{v}_j}{\kappa^2} \tag{4.7}$$

where $\kappa^2 = \kappa_1^2 + \kappa_2^2 + \kappa_3^2$. Therefore,

$$\frac{d\hat{v}_i}{d\tau} = 2\Gamma\epsilon_{nm3}\left(\frac{\kappa_i\kappa_n}{\kappa^2} - \delta_{in}\right)\hat{v}_m. \tag{4.8}$$

Statistical quantities describing homogeneous turbulence are found by averaging over an ensemble of such flows and allowing the box size to become infinite. In particular, the spectrum tensor is given by

$$E_{ij}(\underline{\kappa}, \tau) = \lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^3 < \hat{v}_i(\underline{\kappa}, \tau)\hat{v}_j^*(\underline{\kappa}, \tau) > . \tag{4.9}$$

Note that by continuity

$$\kappa_i E_{ij} = 0 \qquad \kappa_j E_{ij} = 0 \tag{4.10}$$

and that by its definition

$$E_{ij}(-\underline{\kappa}, \tau) = E_{ji}(\underline{\kappa}, \tau). \tag{4.11}$$

The dynamical equation for the spectrum tensor is formed from (4.8), and is

$$\frac{dE_{ij}}{d\tau} = 2\Gamma\epsilon_{nm3}\left[\left(\frac{\kappa_i\kappa_n}{\kappa^2} - \delta_{in}\right)E_{mj} + \left(\frac{\kappa_j\kappa_n}{\kappa^2} - \delta_{jn}\right)E_{im}\right]. \tag{4.12}$$

It is important to note that in RDT the spectrum equation is a *closed* equation.

The turbulent stresses are given by

$$R_{ij}(\tau) = \overline{v_i v_j} = \int E_{ij}(\underline{\kappa}, \tau)d^3\underline{\kappa}. \tag{4.13}$$

The evolution equation for $R_{ij}$, obtained by integrating (4.12) over wavenumber space, is

$$\frac{dR_{ij}}{d\tau} = 2\Gamma\epsilon_{nm3}(M_{mjin} + M_{mijn} - \delta_{in}R_{mj} - \delta_{jn}R_{mi}) \tag{4.14a}$$

where

$$M_{ijpq}(\tau) = \int \frac{\kappa_p \kappa_q}{\kappa^2} E_{ij}(\underline{\kappa}, \tau) d^3\underline{\kappa}. \tag{4.14b}$$

Note that the $R_{ij}$ equations are *not* closed since they contain the **M** terms, which arise from the pressure.

*4.2 Permanence of the energy spectrum*

It is immediately evident from (4.12) that

$$\frac{dE_{ii}}{d\tau} = 0. \tag{4.15}$$

Thus, rapid rotation (without strain) does not alter the distribution of energy among the Fourier modes; therefore, one should not expect a Taylor-Proudman reorganization of the flow into two-dimensional turbulence. Since the energy spectrum is constant, the structure tensor $Y_{ij}$ is also constant (in the rotating frame).

*4.3 Initially isotropic turbulence*

If at $\tau = 0$ the spectrum is isotropic, then

$$E_{ij}(\underline{\kappa}, 0) = \frac{E(\kappa)}{4\pi\kappa^2}\Big(\delta_{ij} - \frac{\kappa_i \kappa_j}{\kappa^2}\Big) \tag{4.16}$$

and (4.12) reduces to $dE_{ij}/dt = 0$. Therefore, *rapid rotation will not change the spectrum tensor of isotropic turbulence* (in the wavenumber range for which RDT is valid). See Speziale *et al.* (1987).

*4.4 Material indifference to rotation*

If the turbulence is two-dimensional (2D) with its axis of independence aligned with the mean rotation (here $\xi_3$), then one may define a stream function such that

$$v_1 = \frac{\partial \psi}{\partial \xi_2} \qquad v_2 = -\frac{\partial \psi}{\partial \xi_1} \tag{4.17a}$$

and the Coriolis terms then put in a modified pressure,

$$p* = p - 2\psi\rho\Gamma. \tag{4.17b}$$

The equations of motion in the rotating frame are then independent of the rotation rate $\Gamma$ (in the full Navier-Stokes equations as well as in RDT). The only effect of the mean rotation is then to rotate the turbulence, and so under the RDT approximation there is no change in the turbulence field as viewed in the rotating frame. Under this very special condition the turbulence is said to be *materially indifferent to rotation* (Speziale 1981).

*4.5 Response of 2-D turbulence to rotation*

The situation is quite different if the turbulence is 2-D with its axis of independence perpendicular to the axis of mean rotation. Consider the situation where the turbulence in independent of $\xi_1$, for which one can define a stream function $\psi$ such that

$$v_2 = \frac{\partial \psi}{\partial \xi_3} \qquad v_3 = -\frac{\partial \psi}{\partial \xi_2}. \tag{4.18a, b}$$

The RDT equations simplify to

$$\frac{\partial v_1}{\partial \tau} = -2\Gamma \frac{\partial \psi}{\partial \xi_3} \tag{4.19a}$$

$$\frac{\partial}{\partial \tau}\Big(\frac{\partial^2 \psi}{\partial \xi_2 \partial \xi_2} + \frac{\partial^2 \psi}{\partial \xi_3 \partial \xi_3}\Big) = 2\Gamma \frac{\partial v_1}{\partial \xi_3}. \tag{4.19b}$$

The Fourier amplitudes are then governed by

$$\frac{d\hat{v}_1}{dt} = 2i\Gamma\kappa_3\hat{\psi} \qquad \frac{d\hat{\psi}}{dt} = 2i\Gamma\frac{\kappa_3}{\kappa^2}\hat{v}_1. \tag{4.20a, b}$$

We will develop the solutions for two cases where the initial turbulence is isotropic in planes perpendicular to $\xi_1$. In both cases the structure anisotropy tensor is

$$y_{ij} = \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/6 \end{pmatrix}. \tag{4.21}$$

The initial values of the stress anisotropies are different in the two cases.

First, consider the case where the initial turbulence has only one component (1C); $\psi$ is everywhere zero and the initial $v_1$ is $v_{10}$. The solutions are

$$\hat{v}_1 = \hat{v}_{10}\cos\Big(2\frac{\kappa_3}{\kappa}\Gamma\tau\Big) \qquad \hat{\psi} = i\frac{\hat{v}_{10}}{\kappa}\sin\Big(2\frac{\kappa_3}{\kappa}\Gamma\tau\Big). \tag{4.22a, b}$$

The turbulent stresses are computed by integrating over all wavenumbers $(\kappa_2, \kappa_3)$, for example

$$R_{11}(\tau) = \int |\hat{v}_{10}|^2 \cos^2\Big(2\frac{\kappa_3}{\kappa}\Gamma\tau\Big) d\kappa_2 d\kappa_3. \tag{4.23}$$

Carrying out the integration in cylindrical coordinates,

$$R_{11}(\tau) = \int_0^\infty |\hat{v}_{10}|^2 \kappa d\kappa \int_0^{2\pi} \cos^2(2\Gamma\tau\sin\theta) d\theta. \tag{4.24}$$

At $\tau = 0$ the $\theta$ integration gives $2\pi$; since $R_{11}(0) = q_0^2$, the first integral is $q_0^2/(2\pi)$. The second integral is expressible in terms of Bessel functions; the result is

$$R_{11}(\tau) = \frac{q_0^2}{2}[1 + \mathrm{J}_0(4\Gamma\tau)]. \tag{4.25a}$$

Similarly,

$$R_{22}(\tau) = \frac{q_0^2}{4}[1 - \mathrm{J}_0(4\Gamma\tau) + \mathrm{J}_2(4\Gamma\tau)] \tag{4.25b}$$

$$R_{33}(\tau) = \frac{q_0^2}{4}[1 - \mathrm{J}_0(4\Gamma\tau) - \mathrm{J}_2(4\Gamma\tau)]. \tag{4.25c}$$

Note that $\mathrm{J}_0(0) = 1$ and $\mathrm{J}_2(0) = 0$, and that both Bessel functions oscillate with decreasing amplitude as time progresses. The off-diagonal stresses, initially zero, oscillate a bit and then return to zero. We see that that rotation converts this 1C-2D turbulence into 3C-2D turbulence, and that eventually an asymptotic state is reached where $R_{11} = 2R_{22} = 2R_{33}$, corresponding to

$$b_{ij} = \begin{pmatrix} 1/6 & 0 & 0 \\ 0 & -1/12 & 0 \\ 0 & 0 & -1/12 \end{pmatrix} = -y_{ij}/2. \tag{4.26}$$

The corresponding mean-square vorticity components are

$$\overline{\omega_1^2}(\tau) = \frac{\overline{\omega_0^2}}{2}[1 - \mathrm{J}_0(4\Gamma\tau)] \tag{4.27a}$$

$$\overline{\omega_2^2}(\tau) = \frac{\overline{\omega_0^2}}{4}[1 + \mathrm{J}_0(4\Gamma\tau) - \mathrm{J}_2(4\Gamma\tau)] \tag{4.27b}$$

$$\overline{\omega_3^2}(\tau) = \frac{\overline{\omega_0^2}}{4}[1 + \mathrm{J}_0(4\Gamma\tau) + \mathrm{J}_2(4\Gamma\tau)]. \tag{4.27c}$$

Note that the vorticity is also redistributed by the rotation and becomes 3C.

In the second case, $v_1$ is zero initially, and the initial vorticity is all in the $\omega_1$ component. The solutions are

$$\hat{v}_1 = i\kappa\hat{\psi}_0 \sin\Big(2\frac{\kappa_3}{\kappa}\Gamma\tau\Big) \tag{4.28a}$$

$$\hat{\psi} = \hat{\psi}_0 \cos\Big(2\frac{\kappa_3}{\kappa}\Gamma\tau\Big). \qquad (4.28b)$$

The turbulent stresses are then

$$R_{11}(\tau) = \frac{q_0^2}{2}[1 - J_0(4\Gamma\tau)] \qquad (4.29a)$$

$$R_{22}(\tau) = \frac{q_0^2}{4}[1 + J_0(4\Gamma\tau) - J_2(4\Gamma\tau)] \qquad (4.29b)$$

$$R_{33}(\tau) = \frac{q_0^2}{4}[1 + J_0(4\Gamma\tau) + J_2(4\Gamma\tau)]. \qquad (4.29c)$$

Note that again the flow becomes 2D-3C, and that the asymptotic state is the same as for case 1. The mean-square vorticity components are

$$\overline{\omega_1^2}(\tau) = \frac{\overline{\omega_0^2}}{2}[1 + J_0(4\Gamma\tau)] \qquad (4.30a)$$

$$\overline{\omega_2^2}(\tau) = \frac{\overline{\omega_0^2}}{4}[1 - J_0(4\Gamma\tau) + J_2(4\Gamma\tau)] \qquad (4.30b)$$

$$\overline{\omega_3^2}(\tau) = \frac{\overline{\omega_0^2}}{4}[1 - J_0(4\Gamma\tau) - J_2(4\Gamma\tau)]. \qquad (4.30c)$$

Note that the vorticity, which was initially all in one component, is redistributed, with the final state (as in the previous case) being

$$\overline{\omega_1^2} = 2\overline{\omega_1^2} = 2\overline{\omega_1^2} = \overline{\omega_0^2}/2. \qquad (4.31)$$

If the turbulence is two-dimensional with the axis of independence at an angle of $\phi$ with respect to the axis of rotation, the components of motion in a frame aligned with the axis of uniformity are as developed above, but with $\Gamma$ replaced by $\Gamma \sin\phi$.

## *4.6 Model spectrum for weakly anisotropic turbulence*

In order to handle RDT for a more general case we need an initial spectrum for anisotropic turbulence. We assume that $E_{ij}(\mathbf{k})$ can be expressed in terms of the tensors $\delta_{ij}$, $k_i$, and $b_{ij}$. For weakly anisotropic turbulence, where only linear terms in $\mathbf{b}$ are retained, the most general form with the required symmetry is

$$E_{ij} = C_1\delta_{ij} + C_2\frac{k_ik_j}{k^2} + C_3 b_{ij} + C_4\frac{b_{in}k_nk_j + b_{jn}k_nk_i}{k^2}$$
$$+C_5\frac{b_{nm}k_nk_m}{k^2}\delta_{ij} + C_6\frac{b_{nm}k_nk_mk_ik_j}{k^4} + O(\mathbf{b}^2). \qquad (4.32)$$

Here the coefficients $C_n$ are functions of $k$, the magnitude of $\mathbf{k}$. For larger anisotropy, additional terms are needed and the coefficients must be allowed to depend on all the invariants formed from $\mathbf{k}$ and $\mathbf{b}$.

The continuity equation requires $k_jE_{ij} = 0$. When applied to (4.31), one finds

$$C_1 + C_2 = 0 \qquad (4.33a)$$

$$C_3 + C_4 = 0 \qquad (4.33b)$$

$$C_4 + C_5 + C_6 = 0. \qquad (4.33c)$$

In order for the last two equations to hold for all $k$, $C_3$-$C_6$ must vary in the same way with $k$, so we can take $C_5 = \alpha C_3$. The resulting spectrum is

$$E_{ij}(\mathbf{k}) = C_1\Big(\delta_{ij} - \frac{k_ik_j}{k^2}\Big)$$
$$+C_3\Big[b_{ij} - \frac{b_{in}k_nk_j + b_{jn}k_nk_i}{k^2} + \frac{b_{nm}k_nk_mk_ik_j}{k^4}$$
$$+\alpha\Big(\frac{b_{nm}k_nk_m}{k^2}\delta_{ij} - \frac{b_{nm}k_nk_mk_ik_j}{k^4}\Big)\Big]. \qquad (4.34)$$

Some helpful relations are

$$\int k^2C(k)d^3\mathbf{k} = \int_{k=0}^{\infty}\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} C(k)k^2\sin\phi\, d\phi\, d\theta\, dk$$
$$= 4\pi\int_0^{\infty} k^2C(k)dk = I \qquad (4.35a)$$

$$\int C(k)\frac{k_ik_j}{k^2}d^3\mathbf{k} = \frac{1}{3}I\delta_{ij} \qquad (4.35b)$$

$$\int C(k)\frac{k_ik_jk_nk_m}{k^4}d^3\mathbf{k} = \frac{1}{15}I\Delta_{ijnm} \qquad (4.35c)$$

$$\Delta_{ijnm} = \delta_{ij}\delta_{nm} + \delta_{in}\delta_{jm} + \delta_{im}\delta_{jn} \qquad (4.35d)$$

$$\int C(k)\frac{k_ik_jk_nk_mk_pk_q}{k^6}d^3\mathbf{k} = \frac{1}{105}I\Delta_{ijnmpq} \qquad (4.35e)$$

$$\Delta_{ijnmpq} = \delta_{ij}\Delta_{nmpq} + \delta_{in}\Delta_{jmpq}$$
$$+\delta_{im}\Delta_{jnpq} + \delta_{ip}\Delta_{jnmq} + \delta_{iq}\Delta_{jnmp}. \qquad (4.35f)$$

From the definitions,

$$\int E_{ij}d^3\mathbf{k} = R_{ij} = q^2(\delta_{ij}/3 + b_{ij}). \qquad (4.36)$$

Integrating (4.34) and comparing with (4.36), one finds

$$I_1 = \frac{1}{2}q^2 \qquad I_3 = \frac{15}{7-2\alpha}q^2. \qquad (4.37)$$

Note that the spectrum model still contains one undetermined constant $\alpha$. The structure anisotropy tensor $y_{ij}$ can be calculated from the model spectrum. One finds

$$y_{ij} = \frac{4\alpha - 2}{7 - 2\alpha}b_{ij}. \qquad (4.38)$$

On comparison of the $M_{ijpq}$ resulting from this spectrum with that resulting from RDT for distortion of isotropic turbulence by arbitrary irrotational strain, one finds that $\alpha = 3/2$ matches RDT. A value $\alpha = 1/2$, which produces $y_{ij} = 0$, gives a rapid pressure-strain model that is almost exactly that used by Launder, Reece, and Rodi (1975).

For sufficiently large $\mathbf{b}$ (4.36) becomes an unrealizable spectrum in some regions of $\mathbf{k}$ space. T.S. Shih has worked out the range of realizability as a function of the invariants of $\mathbf{b}$. The realizable range is very small for $\alpha = 3/2$, and appears to be largest for $\alpha = 1/2$, where the structure tensor is isotropic.

## *4.7 RDT for rotation of weakly isotropic turbulence*

In this section we outline the RDT analysis for rotation of weakly isotropic turbulence. Although we could solve this problem by solving the coupled spectrum tensor equations, more insight to the physics is provided by solving for the Fourier coefficients. Seeking solutions of the form

$$\hat{v}_\alpha(\underline{\kappa}, \tau) = a_\alpha \exp(i\beta\tau) \qquad (4.39)$$

(4.8) gives

$$i\beta a_1 - 2\Gamma\frac{\kappa_1}{\kappa^2}(a_2\kappa_1 - a_1\kappa_2) + 2\Gamma a_2 = 0 \qquad (4.40a)$$

$$i\beta a_2 - 2\Gamma\frac{\kappa_2}{\kappa^2}(a_2\kappa_1 - a_1\kappa_2) - 2\Gamma a_1 = 0 \qquad (4.40b)$$

$$i\beta a_3 - 2\Gamma\frac{\kappa_3}{\kappa^2}(a_2\kappa_1 - a_1\kappa_2) = 0. \qquad (4.40c)$$

This linear equation system has non-trivial solutions only if the determinant of the coefficient matrix vanishes. This condition gives

$$\beta^2 = 4\Gamma^2\Big(1 - \frac{\kappa_1^2 + \kappa_2^2}{\kappa^2}\Big) = 4\Gamma^2\frac{\kappa_3^2}{\kappa^2} > 0. \tag{4.41}$$

Note that the solutions are *undamped oscillations* at frequency $\beta(\underline{\kappa})$.

The solution for $\kappa_3 = 0$ is

$$\hat{v}_1(\underline{\kappa},\tau) = \hat{v}_1(\underline{\kappa},0) - C(\underline{\kappa})\kappa_2\tau \tag{4.42a}$$

$$\hat{v}_2(\underline{\kappa},\tau) = \hat{v}_2(\underline{\kappa},0) + C(\underline{\kappa})\kappa_1\tau \tag{4.42b}$$

$$\hat{v}_3(\underline{\kappa},\tau) = \hat{v}_3(\underline{\kappa},0) \tag{4.42c}$$

where

$$C = 2\Gamma\Big(\frac{\kappa_1\hat{v}_1(\underline{\kappa},0) + \kappa_2\hat{v}_2(\underline{\kappa},0)}{\kappa^2}\Big). \tag{4.42d}$$

But for $\kappa_3 = 0$ the numerator of $C$ is zero by continuity, and hence the Fourier coefficients of these modes do not change under rapid rotation. Thus, these coefficients can also be regarded as undamped oscillations at frequency $\beta(\underline{\kappa})$.

The solution for the Fourier coefficients is therefore

$$\hat{v}_i = a_{i+}e^{i\beta\tau} + a_{i-}e^{-i\beta\tau}. \tag{4.43}$$

$a_{1\pm}$ and $a_{2\pm}$ are related by (4.40a) or (4.40b),

$$(\pm i\frac{\kappa_3}{\kappa} + \frac{\kappa_1\kappa_2}{\kappa^2})a_{1\pm} = (\frac{\kappa_1^2}{\kappa^2} - 1)a_{2\pm}. \tag{4.44}$$

The coefficients $a_{i\pm}$ are set by the initial values,

$$\hat{v}_{i0} = a_{i+} + a_{i-} \tag{4.45}$$

where $\hat{v}_{i0} = \hat{v}_i(\underline{\kappa},0)$. Using (4.44) and (4.45),

$$a_{1\pm} = \pm i\frac{\kappa}{2\kappa_3}\Big[\Big(\mp i\frac{\kappa_3}{\kappa} + \frac{\kappa_1\kappa_2}{\kappa^2}\Big)\hat{v}_{10} + \Big(1 - \frac{\kappa_1^2}{\kappa^2}\Big)\hat{v}_{20}\Big]. \tag{4.46}$$

The elements of the spectrum tensor can now be computed. For example, one finds

$$E_{11}(\underline{\kappa},\tau) = \frac{\kappa^2}{2\kappa_3^2}\Big\{\Big(\frac{\kappa_3^2}{\kappa^2} + \frac{\kappa_1^2\kappa_2^2}{\kappa^4}\Big)E_{110}$$

$$+\Big(1 - \frac{\kappa_1^2}{\kappa^2}\Big)^2 E_{220} + \frac{\kappa_1\kappa_2}{\kappa^2}\Big(1 - \frac{\kappa_1^2}{\kappa^2}\Big)(E_{120} + E_{210})$$

$$+\Big[\Big(\frac{\kappa_3^2}{\kappa^2} - \frac{\kappa_1^2\kappa_2^2}{\kappa^4}\Big)E_{110} - \Big(1 - \frac{\kappa_1^2}{\kappa^2}\Big)^2 E_{220}$$

$$-\frac{\kappa_1\kappa_2}{\kappa^2}\Big(1 - \frac{\kappa_1^2}{\kappa^2}\Big)(E_{120} + E_{210})\Big]\cos(2\beta\tau)$$

$$-\Big[\frac{2\kappa_1\kappa_2\kappa_3}{\kappa^3}E_{110} + \frac{\kappa_3}{\kappa}\Big(1 - \frac{\kappa_1^2}{\kappa^2}\Big)(E_{120} + E_{210})\Big]\sin(2\beta\tau)\Big\} \tag{4.47}$$

where $E_{ij0} = E_{ij}(\underline{\kappa},0)$.

If the initial turbulence is isotropic with a spectrum given by (4.16), then the coefficients of the sin and cos terms vanish and the constant coefficient becomes $E_{11}(\underline{\kappa},0)$, *i.e.* the spectrum is unchanged, as noted before.

If the initial spectrum is anisotropic, then at each wavenumber $\underline{\kappa}$ the spectrum will oscillate at a frequency $2\beta(\underline{\kappa})$. This might lead one to expect undamped oscillations in the turbulent stresses, but as in the 2D cases treated above the result is instead a damped oscillation of the stresses. Using the initial spectrum (4.34), T.S. Shih carried out the integrations (with the help of the symbolic manipulator MACSYMA) and arrived at equations for the turbulent stresses in terms of the initial anisotropies $b_{ij0}$; for example,

$$R_{11}(\tau) = \frac{q_0^2}{30720(\Gamma\tau)^5}\Big\{(30720b_{110} + 10240)(\Gamma\tau)^5 +$$

$$\frac{15}{7-2\alpha}\Big([3840(b_{110} - b_{220})(\Gamma\tau)^4 - (720b_{220} + 2160b_{110})(\Gamma\tau)^2$$

$$+225b_{220} + 315b_{110}]\sin(4\Gamma\tau) + [1920(b_{110} - b_{220})(\Gamma\tau)^3$$

$$-(900b_{220} + 1260b_{110})\Gamma\tau]\cos(4\Gamma\tau) - 12288b_{110}(\Gamma\tau)^5\Big)\Big\}. \tag{4.48}$$

Note that the RDT predicts the damped oscillations observed in the numerical simulation, and that the asymptotic steady-state solution is

$$R_{11} = q_0^2\Big(b_{110} + \frac{1}{3} - \frac{2}{5}\frac{15}{7-2\alpha}b_{110}\Big). \tag{4.49}$$

Using (4.38), the asymptotic state corresponds to

$$b_{ij} = -y_{ij}/2. \tag{4.50}$$

The same damped oscillation and asymptotic behavior is found for all other turbulent stress tensor components.

## 5. Physical interpretation of the asymptotic state

We have seen that rapid rotation does not change **y** but does alter the stress anisotropy **b**, driving the turbulence to a state where $b_{ij} = -y_{ij}/2$. A physical explanation for this will now be given.

Continuity requires that the Fourier coefficient $\hat{\mathbf{v}}$ be perpendicular to the vector $\underline{\kappa}$. The anisotropy in strained flows is due in part to redistribution of energy to different wavenumbers and in part to preferential alignment of the $\hat{\mathbf{v}}$ vectors. Rotation causes the $\hat{\mathbf{v}}$ vector (real and imaginary parts) to precess around the wavenumber vector (Fig. 3) without change in magnitude (under RDT approximations). In the notation of Fig. 3,

$$\hat{v}_1 = \hat{\mathbf{v}}(-\cos\alpha\cos\phi\cos\theta - \sin\alpha\sin\theta) \tag{5.1a}$$

$$\hat{v}_2 = \hat{\mathbf{v}}\cos\alpha\sin\phi \tag{5.1a}$$

$$\hat{v}_3 = \hat{\mathbf{v}}(-\cos\alpha\cos\phi\sin\theta + \sin\alpha\cos\theta) \tag{5.3a}$$

The precession rate is uniform at any given $\underline{\kappa}$, which means that there is equal probability of finding the $\hat{\mathbf{v}}$ vector at any angle $\alpha$. Therefore, the averages (time or ensemble) of $\sin^2\alpha$ and $\cos^2\alpha$ are $1/2$, and so

$$b_{11} = \frac{1}{q^2}\int|\hat{\mathbf{v}}|^2\Big(\frac{1}{2}\cos^2\phi\cos^2\theta + \frac{1}{2}\sin^2\theta] - \frac{1}{3}\Big)d^3\underline{\kappa}. \tag{5.4}$$

Similarly, the structure anisotropy $y_{11}$ is

$$y_{11} = \frac{1}{q^2}\int|\hat{\mathbf{v}}|^2\Big(\sin^2\phi\cos^2\theta - \frac{1}{3}\Big)d^3\underline{\kappa}. \tag{5.5}$$

That $b_{11} = -y_{11}/2$ is evident writing (5.4) and (5.5) entirely in terms of cosines. The same relationship holds for all other components. Hence, the asymptotic relationship $b_{ij} = -y_{ij}/2$ will hold for any turbulence in which the phase of the Fourier vector is random.

Thus, the primary effect of the rotation is to *randomize the phase* of the Fourier coefficients. This randomization reduces the anisotropy of the turbulent stresses (a linear effect) in addition, it reduces the non-linear spectral transfer and hence reduces the dissipation rate (Bardina *et al.* 1985).

## 6. A simple turbulence model for rapid rotation

We have explored a number of possibilities for modifying the rapid pressure-strain model. One can consider allowing **M** to depend both on **b** and **y**. Over 100 linearly independent tensors could appear in such a model; the linear version contains no unknown coefficients but does not display the damped oscillations, and the complete model would be very complex. Another alternative is to include the evolution equations for $\mathbf{M}$, which would increase the number of dependent variables by 36. This approach requires a model for

$$N_{ijpqrs} = \int \frac{k_p k_q k_r k_s}{k^4} E_{ij}(\mathbf{k}) d^3\mathbf{k}. \qquad (6.1)$$

Although this approach is probably not practical either, some exploration of this idea was carried out by the author and T.S. Shih to see if it might work. We modeled $\mathbf{N}$ in terms of $\mathbf{M}$, retaining only linear terms. There are sixteen linearly independent tensors that must be involved, and after using continuity and definitional constraints there is one undetermined coefficient remaining. We used this in a model of RDT, attempting to match the RDT solutions. The model solutions did display something resembling damped oscillations at the correct frequency, but did not level out to an asymptotic steady state.

A much simpler idea seems far more practical. A model displaying the essential characteristics discussed above is (in inertial coordinates)

$$\dot{b}_{ij} = (2X-1)(b_{ik}\Omega_{kj} + b_{jk}\Omega_{ki})$$
$$-\beta X \Omega (2b_{ij} + y_{ij}) \qquad (6.1a)$$

$$X = \sqrt{\frac{2\Omega_n\Omega_m(y_{nm} + \delta_{nm}/3)}{\Omega^2}} \qquad (6.1b)$$

where $\Omega^2 = \Omega_n\Omega_n$ and $\Omega_i$ is the mean vorticity. The $X$ term assures that rotation has only a kinematic effect on the turbulence when the turbulence is 2D with its axis of independence aligned with the mean vorticity (material indifference), and tempers the effect of rotation as the sin of the angle between these axes. The $\beta$ term produces the proper asymptotic state. For the 2D cases treated above, $X = 1$ and the equations in the rotating frame are

$$\frac{db_{11}}{d\tau} = -4\Gamma b_{12} - 2\beta\Gamma(2b_{11} - 1/3) \qquad (6.2a)$$

$$\frac{db_{22}}{d\tau} = +4\Gamma b_{12} - 2\beta\Gamma(2b_{22} + 1/6) \qquad (6.2b)$$

$$\frac{db_{12}}{d\tau} = 2\Gamma(b_{11} - b_{22}) - 2\beta\Gamma b_{12}. \qquad (6.2c)$$

The solutions are of the form

$$b_{ij} = e^{-2\beta\Gamma\tau}[A\cos(4\Gamma\tau) + B\sin(4\Gamma\tau)] - y_{ij}/2. \qquad (6.3)$$

Comparison with the exact solutions for these cases suggests $\beta = 0.35$. Incorporation of this model in more comprehensive turbulence models is now being explored.


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Dr. N.N. Mansour provided the numerical simulations that stimulated this work and numerous other useful comments. Dr. T.S. Shih extended the author's development of the RDT problem and carried out the most complicated RDT integrations. The author is indebted to the conference organizers for their invitation and patience.


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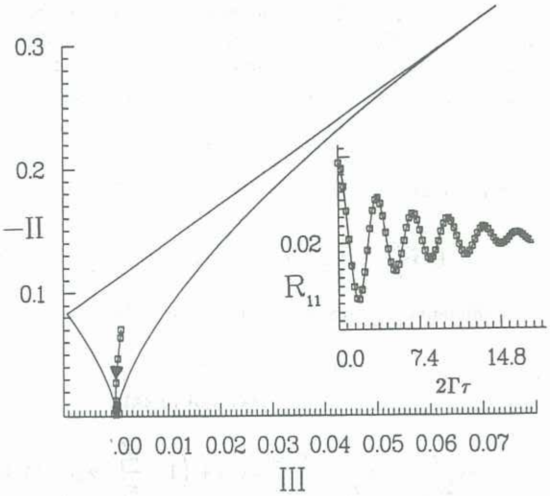


Fig. 1 Typical numerical simulation of the rotation of anisotropic homogeneous turbulence (by N.N. Mansour).

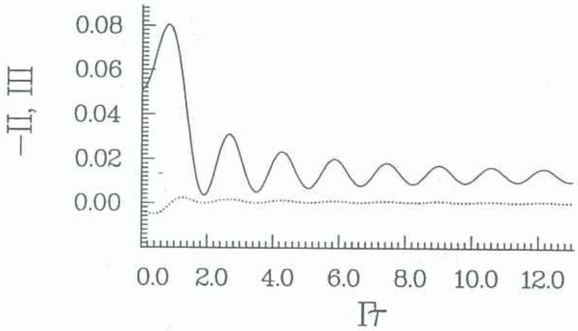


Fig. 2 Typical RDT solution for the rotation of initially anisotropic homogeneous turbulence (by T.S. Shih).

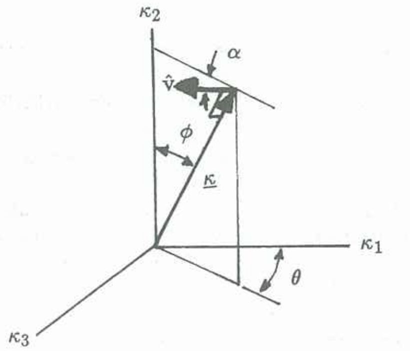


Fig. 3 Wavenumber space.