MIXING IN DENSITY-STRATIFIED CONJUGATE FLOWS

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ABSTRACT

Conjugate two-layer stratified shear flows have been studied in the laboratory using combined laser-induced fluorescence and laser-Doppler velocimetry. Use of this high-resolution, non-intrusive instrumentation has enabled a new evaluation of the internal hydraulic jump, the entrainment associated with it, and the role of upstream and downstream controls in determining the flow structure.

INTRODUCTION

Internal hydraulic jumps are encountered in many flow systems of interest to engineers and environmental scientists. Such jumps occurring in nature have been observed in Föhn winds, katabatic flows, and fjords (Schweitzer, 1953; Ball, 1959; Lied, 1964; Clarke et al, 1981; Forman and Denton, 1985). In an engineering context, they can occur in discharge channels where the density of the effluent differs from the ambient fluid density, and it is usually the mixing within the channel that is of primary concern.

We consider a dense fluid discharged at a steady rate at the bottom of the upstream end of a rectangular mixing channel, as in Figure 1 (a). The channel is horizontal and connected to a receiving body containing a finite depth of water with a uniform lower density. The rate of outflow from the channel of this denser lower layer is controlled at the downstream end by either a free overfall or a broad-crested weir. The discharged and ambient fluids are miscible and their relative density difference is small (less than 2%).

The complexity of such internal flows has induced many previous investigators to make various simplifying assumptions to aid in their analysis. The most appealing of these is to imagine that each layer of the flow has a uniform velocity and density distribution (*e.g.*, Baddour, 1987) and introduce a one-dimensional "hydraulic" theory. However, few experimental data are available to support such a one-dimensional analysis. The studies reported here were planned specifically to evaluate this approach and provide a definitive statement of the role of the upstream and downstream flow controls.

PRELIMINARY OBSERVATIONS

Laboratory observations have shown that there are four substantially different configurations that can occur in a density-stratified channel flow. They are illustrated in Figures 1(a), 1(b), 1(c) and 1(d).

The most obvious and usual flow field is depicted in Figure 1(a), in which the dense lower fluid is supplied at a rate that induces a jet-like flow in the lower layer immediately downstream of the source. A second flow control, provided at a downstream overfall, forces an internal hydraulic jump transition between upstream and downstream flows.

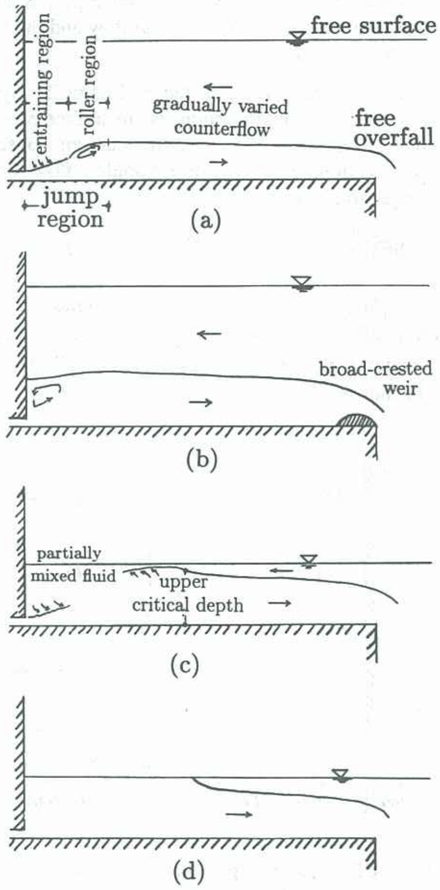

Figure 1. Four channel flow regimes identified (a) free internal hydraulic jump (b) flooded jump (c) critical upper flow (d) blocked flow.

When the downstream control is raised in elevation, the configuration in Figure 1(b) is established in which the jump advances upstream and floods the jet. These two flows are analogous to the corresponding well-known and exhaustively-studied open channel flows, except that here entrainment occurs from the upper fluid layer into the lower.

A third situation, as depicted in Figure 1(c), occurs when the depth of the upper layer is reduced to the point that the jet entrainment creates sufficient flow velocity within the upper layer that a flow control is estabilished in the upper layer. A fourth condition, Figure 1(d), occurs when the lower flow blocks the upper flow and an intrusive wedge is created in the layer of upper fluid, a configuration frequently seen in estuaries.

ANALYSIS AND EQUATIONS OF MOTION

When a layer of dense fluid flows beneath a less dense fluid and both layers are sufficiently shallow for fluid velocities to be approximately uniform over the depth of each layer, it can be shown (Schijf and Schönfeld, 1953) that the velocity of propagation, c, of long internal waves is given by

$$c = \frac{u_1 h_2 + u_2 h_1}{h_1 + h_2} \pm \sqrt{\frac{g' h_1 h_2}{h_1 + h_2} - \frac{h_1 h_2}{(h_1 + h_2)^2}(u_1 - u_2)^2} \quad (1)$$

where h_1 and h_2 are the depths of the two layers, u_1 and u_2 are their velocities and $g' = g(\rho_1 - \rho_2)/\rho_2$, with ρ_1 and ρ_2 being the respective densities of the lower and upper fluids. When c=0 then this equation can be rearranged to the form

$$F_1^2 + F_2^2 = 1 \quad (2)$$

where $F_1 = u_1/\sqrt{g' h_1}$ and $F_2 = u_2/\sqrt{g' h_2}$ are the densimetric Froude numbers of the two layers. When Eq.(2) is satisfied, the two layer flow is said to be a critical flow and the location is referred to as a control.

Now consider the flow depicted in Figure 2. By taking due account of the stagnation point rise in surface elevation at section 0, and applying mass and momentum conservation principles at sections 0 and 1, a dimensionless flow force conservation equation is obtained (Rasi, 1989).

$$2SF_0^2\left[\frac{S^2}{r_1} + \left(\frac{S-1}{H-r_1}\right)^2\left(\frac{H}{2} - 1\right) - 1\right] = S - r_1^2 \quad (3)$$

where $r_1 = h_1/h_0$, is the depth ratio after the jump; $S = q_1/q_0$, is the dilution after the jump; $H = d/h_0$, is the dimensionless ambient water depth; $F_0 = q_0/\sqrt{g'_0 h_0^3}$, is the discharge densimetric Froude number.

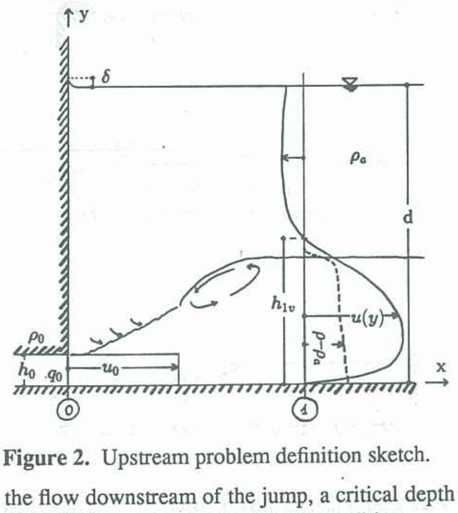

Figure 2. Upstream problem definition sketch.

For the flow downstream of the jump, a critical depth flow at a free overfall downstream imposes the condition

$$SF_0^2\left[\frac{S^2}{r_c^3} + \frac{(S-1)^2}{(H-r_c)^3}\right] = 1 \quad (4)$$

where $r_c = h_c/h_0$ is the lower layer critical depth ratio.

The gradually varied sub-critical flow between the jump and the critical control section can be shown to satisfy the equation (Rasi,1989)

$$\frac{dr}{d\xi} = \frac{SF_0^2\left[\left(f_b + \frac{2r}{B}f_{wl}\right)\frac{S^2}{r^3} + f_i\left(\frac{S}{r} + \frac{S-1}{H-r}\right)^2\left(\frac{1}{r} + \frac{1}{H-r}\right) + \frac{2f_{wu}}{B}\left(\frac{S-1}{H-r}\right)^2\right]}{8\left[1 - SF_0^2\left(\frac{S^2}{r^3} + \frac{(S-1)^2}{(H-r)^3}\right)\right]} \quad (5)$$

where $r = h(x)/h_0$ is the dimensionless lower layer depth; $\xi = x/h_0$ is the dimensionless distance from the downstream control; $B = b/h_0$ is the dimensionless channel width.

In the derivation of Eq.(5), it has been assumed that turbulent boundary friction is represented by a Darcy-Weisbach formulation where f_b, f_i, f_{wl}, and f_{wu}, are the friction factors at the channel bottom, fluid interface, and lower and upper layer channel walls respectively. It is recognized that f_i is simply an empirical representation of the vertical momentum transfer mechanisms at the interface, including waves and entrainment. The values of S and r_1 must satisfy the obvious physical constraints $0 \le r_1 \le H$ and $S \ge 1$ together with

$$SF_0^2\left[\frac{S^2}{r_1^3} + \frac{(S-1)^2}{(H-r_1)^3}\right] \le 1 \quad (6)$$

$$SF_0^2\left[\frac{S^2}{r_1} + \left(\frac{S-1}{H-r_1}\right)^2\right] \le 2(1 - r_1) + F_0^2 \quad (7)$$

which state respectively that the flow downstream of the jump must be sub-critical (Harleman, 1960), and that the specific energy cannot increase across the jump (Baddour and Abbink, 1983).

For any specified value of S, Eq.(3) can be solved for r_1 if H and F_0 are fixed. Only one real root satisfies constraint (7). Note that if S = 1 then Eq.(3) reduces to

$$r_1 = \frac{1}{2}\left(\sqrt{1 + 8F_0^2} - 1\right) \quad (8)$$

the result for immiscible jumps given by Yih and Guha (1955). Baddour and Abbink (1983) show that for $H \to \infty$ there is a maximum dilution achievable

$$S_m = \frac{2}{3}F_0^{2/3}\left[1 + 1/2F_0^2\right] \quad (9)$$

The gradually varied flow Eq. (5) can be integrated upstream of the downstream control, if values of S and F_0 are assumed along with friction factors. The integration begins at $\xi = 0$ where $r = r_c$, and r_c is determined from Eq.(4), the criticality condition at this control. The general solution is then obtained by finding the location where the r_1 value determined from the upstream equation matches that found by integrating the downstream equation upstream. The solution technique is shown in Figure 3 for specified values of the initial jet Froude number F = 8.88, dimensionless overall depth H = 26.6, and a defined location of the downstream control.

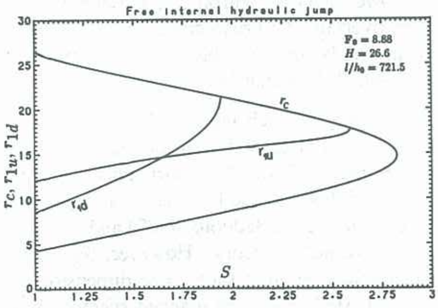

Figure 3. Solving for a free internal jump in the (S, r_1) plane.

The curve labeled r_c is the solution of Eq. (4); that labeled r_{1u} is the solution for r_1 obtained from the conservation of flow force Eq.(3). The curve labeled r_{1d} is the value of r_1 at the jump location determined by integrating Eq.(5) using the prescribed initial condition. The solution values of S and r_1 are defined by the intersection of these two curves.

It is not always possible that a common intersection point will exist for the two r_{1u} and r_{1d} curves. This will occur when either the r_{1u} curve is lowered by reducing F_0, or the r_{1d} curve is raised by either moving the downstream control further downstream, or by raising the sill elevation with a weir. In either case, the internal jump becomes flooded (see Figure 1(b)).

Another possibility occurs if F_0 is increased to produce the opposite effect, in which case the r_{1u} expected from flow force conservation is always higher than the r_{1d} expected from the downstream gradually-varied flow integration. When this occurs another critical point, which becomes a downstream control for flow in the upper layer, is created immediately downstream of the jump. This corresponds to Figure 1(c).

Finally, when F_0 is so large that the entire depth of the two layer flow d is occupied by source fluid, then the dilution S = 1 and the condition in Figure 1(d) results. r_c is specified and the profile is integrated until it meets the free surface. This can occur if the downstream control is far enough from the source.

EXPERIMENTAL ANALYSIS AND METHODS

A stratified flow was constructed in the laboratory as shown in Figure 4. The experiments were designed to provide detailed information about the uniformity of the density and velocity distributions in each flowing layer, since this is a key assumption in the anaylsis. Horizontal velocities were measured using a laser-Doppler velocimeter system and a density-stratified flow that was optically homogeneous, as described in Hannoun and List (1988). Vertical density profiles were obtained with a laser-induced fluorescence system for concentration measurement that profiles an instantaneous record of dye concentration along the length of a directed laser beam. Comprehensive details of this equipment are provided in other publications (Hannoun and List, 1988; Papanicolaou and List, 1988; Papantoniou and List, 1989). Typical velocity and density profiles in the flow established with this equipment are shown in Figures 5 and 6 respectively.

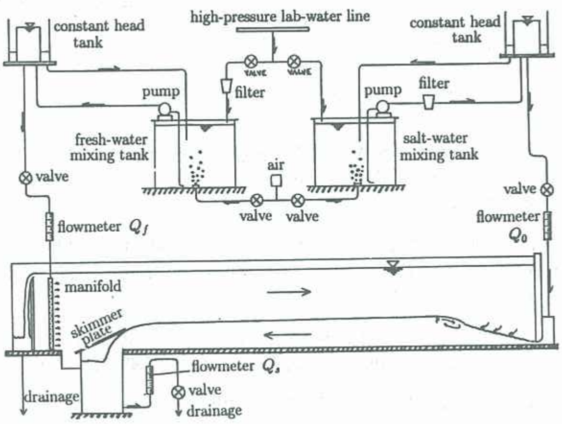

Figure 4. Laboratory set up for internal flow experiments.

A system was developed to establish the dilution of the discharged flow. This involved constructing a flow skimming device at the downstream control as shown in Figure 4. By coloring the bottom layer of fluid with an intense blue dye, it was possible to set the skimmer such that no blue fluid passed over the skimmer or no clear fluid passed below it. The rate of accumulation of flow below the skimmer provided exceptionally reproducible measurements of the dilution S.

RESULTS

Experiments were performed over a range of densimetric Froude numbers from 3.2 to 14.5 and dimensionless water depths from 15.3 to 76.6. The experimentally measured dilutions are presented in Figure 7 as functions of the densimetric Froude number. A comparison is also made with the predicted dilution as obtained by the previously discussed theory.

It can be seen that the comparison between theory and prediction is reasonable for the shallower water depths but becomes less agreeable as the water depth and Froude number increase.

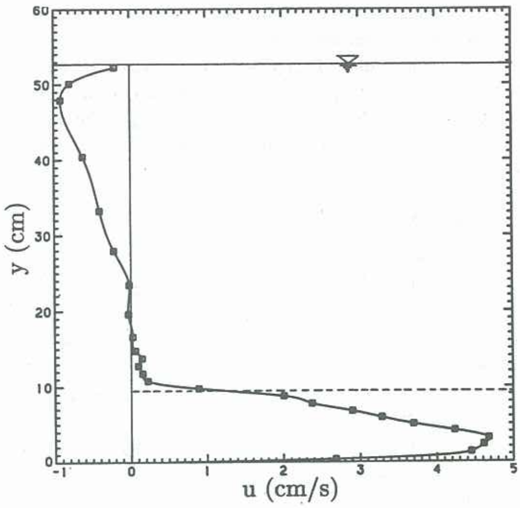

Figure 5. Typical velocity profile in a subcritical flow region

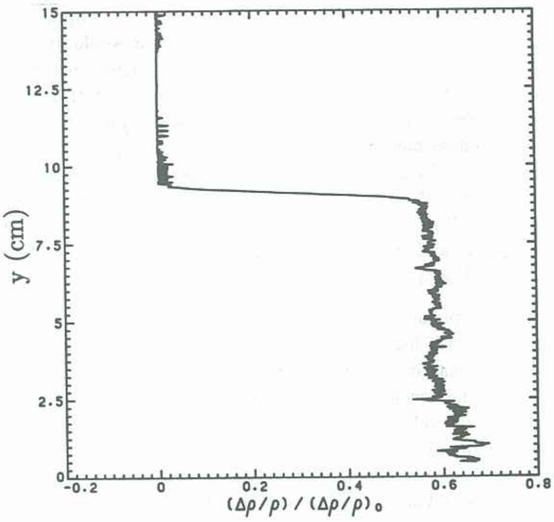

Figure 6. Typical density profile in a subcritical flow region

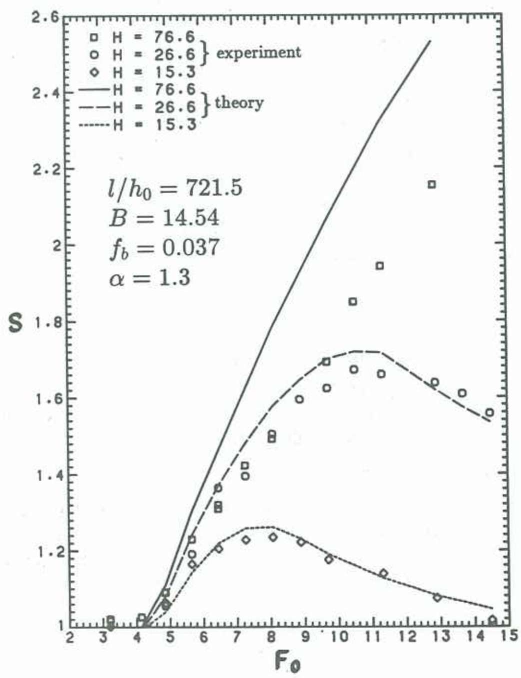

Figure 7. Measured and predicted dilutions.

It is believed that the reason for the lack of agreement is the fact that as F_0 and H increase, the intersection point of the r_{1u} and r_{1d} curves is very sensitive to the location of either curve (see Figure 3). Possible sources of error that would result in the movement of either of these curves include the assumption of uniformity in the profiles of density and velocity, and a lack of information about the length of the jump zone.

DISCUSSION AND CONCLUSIONS

The interesting result to emerge from the analysis and experiments is that the dilutions predicted on the basis of gross mean flow properties and those measured experimentally are in good agreement, as evidenced in Figure 7. This is somewhat surprising since these dilutions are very dependent on the level of entrainment that occurs, and yet no entrainment assumptions are considered in the analysis. In addition to predicting the dilution over a reasonably wide range of densimetric Froude number, the analysis also indicates a significant dependency of this entrainment on the water depth (see Figure 7).

Furthermore, the prediction that there is a maximum dilution that will occur for any given Froude number and water depth is closely followed. The departure from the prediction surprisingly occurs as the water depth becomes large, which one would imagine would be the simpler flow to predict. It is of interest that these predictions are in no way dependent on a consideration of the turbulence properties of the discharge jet. Clearly, the entrainment and limiting dilution must depend indirectly on the collapse of the turbulence in the jet as the local Richardson number approaches the limiting value. A close study of the turbulence and entrainment appears to indicate that the entrainment actually occurs upstream of the jump and not in the roller region that forms the jump; there is in fact very little entrainment within the jump.

One interpretation of the flow is therefore the following. The initial flow, as it leaves the source, behaves as a two dimensional wall jet that entrains the ambient fluid in the usual way. However, since the entrainment both increases the depth of flow and decreases the local mean velocity, the local densimetric Froude number will decrease with distance downstream. Equivalently, the local bulk Richardson number increases. This reduces the entrainment from that which would occur if the flow were a pure wall jet.

The densimetric Froude number continues to reduce with increasing distance downstream as a consequence of the entrainment until it attains a value that is conjugate to the downstream Froude number that is developed from integrating back from the critical depth at the free overfall. In this sense, the dilution and maximum depth of the denser flow are still determined by the distance downstream to the control. The further downstream this control is located the deeper is the subcritical conjugate flow and the lower the downstream Froude number. This in turn requires that the conjugate upstream Froude number be larger and consequently the jump must move upstream. At the point when the depth downstream of the jump reaches a value specified by

$$h_1 = \frac{h_0}{2}\left[\sqrt{1+8F_0^2}-1\right] \qquad (12)$$

then the jump floods the source since the conjugate upstream depth is deeper than any possible jet flow. Alternatively, should the upstream densimetric Froude number decrease to the point where the conjugate depth is lower than the depth prescribed by the downstream control, then the jump will flood the source.

In the situation when the total water depth is limited, two possible different conditions can occur when either the source jet Froude number is increased to the point that the entrainment demand makes the upper flow layer critical, or the water profile integrated from downstream intersects the water surface.

All of these modes of operation can be defined on a dimensionless water depth H versus Froude number F_0 plot for any specified downstream control location, which is presented for a "long" channel in Figure 8.

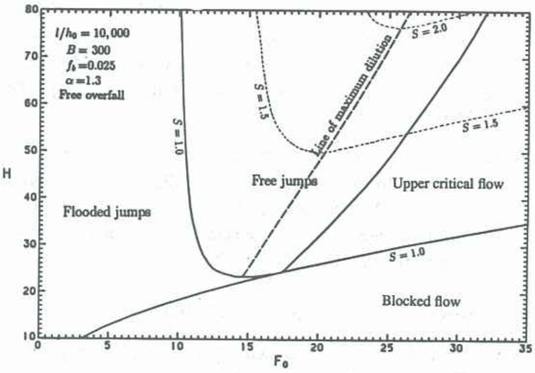

Figure 8. Mixing mode domains for far downstream control.

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