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THE GENERALIZED LAGRANGIAN MEAN EQUATION AND A MECHANISM FOR STREAMWISE VORTICES IN A TURBULENT SHEAR FLOW

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ABSTRACT

The Generalized Lagrangian-mean equations are used to derive evolution equations for the perturbation flow about a turbulent mean base flow which is homogeneous in the streamwise and spanwise directions. They are used to deduce the mechanism which leads to the formation of streamwise vortices in the wall region of turbulent bounded flows.

INTRODUCTION

The presence of streamwise or almost streamwise vortices in both bounded an unbounded turbulent shear flows is well documented, but questions remain about the origin of these vortices. Benny & Lin (1962) first proposed that those in the wall region of a turbulent boundary layer form owing to the non-linear interaction of a two- and a three-dimensional wave, but this mechanism was strongly criticized by Stuart (1967). Coles (1979) proposed, and latter retracted, a Taylor Görtler type instability. And recently Jang et al (1986) have employed direct resonance as a possible explanation for their appearance.

A major difficulty in isolating the mechanism which gives rise to these vortices is in formulating the problem: ideally we should like to be in the reference frame of the mean flow; we can then consider the vortices as a secondary flow about the mean and deduce their evolution with time. But if we attempt this using the Navier Stokes equation, difficulties arise in correctly accounting for how oscillatory disturbances affect the mean field and vice versa. Similar problems have plagued the analysis of some water-wave mean-flow interactions and this lead Andrews and McIntyre (1978) to derive what have come to be known as the Generalized Lagrangian Mean equations. These equations are an *exact* and very general Lagrangian-mean description of the back effect of oscillatory disturbance upon the mean state.

In the present work we employ these equations to study the aetiology of streamwise vortices in turbulent channel flow and show that three ingredients are necessary for their formation: Streamwise shear of sufficient magnitude; an oscillatory flow field; and a weak spanwise distortion of the mean field. The mechanism by which they form is similar to the Craik-Leibovich mechanism by which Langmuir cells form; that is, background, initially randomly oriented vorticity, is oriented by the fluctuating field and amplified in the streamwise direction by the imposed shear.

These vortices cause a local inflexion point in the streamwise velocity profile, but the instability induced by it is initially weak. As the shear acts to intensify the vortices, however, so does the strength of the instability, culminating in an exponential amplification in vortex strength and a violent upthrust of fluid from the wall. This last event has been widely observed and is known as ejection. Throughout the process the vortices rise slowly from the wall.

THE GENERALIZED LAGRANGIAN MEAN EQUATIONS

The Generalized Lagrangian-mean (GLM) equations of Andrews & McIntyre (1978) are an exact and very general Lagrangian-mean description of the back effect of oscillatory disturbances upon the mean state. The Lagrangian-mean velocity so described, however, is not the 'mean following a single fluid particle'; it is the velocity field describing trajectories about which the fluctuating particle motions have zero mean, when any averaging process, be it temporal, spatial, ensemble or other, is applied. The most appropriate choice of average is determined by the problem; for example a time average would be used for time periodic flows. To express ideas like a 'steady mean flow', an Eulerian description of the Lagrangian-mean, with position x and time t as independent variables, is desirable. Hence the GLM description is really a hybrid Eulerian-Lagrangian description of wave mean-flow interactions.

To define an exact Lagrangian-mean operator $\langle\ \rangle^L, \overline{(\ )}^L$, corresponding to any given Eulerian-mean operator $\langle\ \rangle, \overline{(\ )}$, we must define with equal generality an exact, disturbance-associated particle displacement field $\xi(\mathbf{x}, t)$. For any scalar or tensor field, ϕ say, of any rank, it is then possible to write

$$\langle\phi(\mathbf{x},t)\rangle^L = \langle\phi^\xi(\mathbf{x}, t)\rangle$$

where

$$\phi^\xi(\mathbf{x}, t) = \phi(\mathbf{x} + \xi, t)$$

Now provided the mapping

$$\mathbf{x} \rightarrow \mathbf{x} + \xi \tag{1}$$

is invertible, there is, for any given $\mathbf{u}(\mathbf{x}, t)$, a unique 'related velocity field' $\mathbf{v}(\mathbf{x}, t)$, such that when the point $\mathbf{x}$ moves with velocity $\mathbf{v}$ the point $\mathbf{x} + \xi$ moves with the actual fluid velocity $\mathbf{u}^\xi$. Then with the requirements $\langle\xi(\mathbf{x}, t)\rangle = 0$ and $\langle\mathbf{v}(\mathbf{x}, t)\rangle = \mathbf{v}(\mathbf{x}, t)$ it is evident that $\mathbf{v} = \bar{\mathbf{u}}^L$, the Lagrangian-mean velocity. Andrews and McIntyre derive the *exact* equations for GLM motion from the compressible Navier Stokes equations. The GLM momentum and continuity equations are, for homentropic flows of constant density in a non-rotating reference frame:

$$\bar{D}^L(\bar{u}_i^L - p_i) + \langle u_k\rangle^L{}_{,i}(\bar{u}_k^L - p_k) + \Pi_{,i} = -\bar{X}_i^L - \langle\xi_{j,i}X_j^l\rangle \tag{2}$$

$$\Pi_i = \frac{\bar{\varpi}^L}{\rho} + \Phi_i^L - \frac{1}{2}\langle u_j^\xi u_j^\xi\rangle \tag{3}$$

$$\bar{D}^L\tilde{\rho} + \tilde{\rho}\nabla\bullet\bar{\mathbf{u}}^L = 0 \tag{4}$$

Observe that the non-linear forcing of the mean flow is expressed in terms of a vector wave property $\mathbf{p}$, whose ith component is

$$p_i = -\langle\xi_{j,i}u_j^l\rangle$$

The vector $\mathbf{p} = p_i(\mathbf{x}, t)$ is the pseudomomentum per unit mass and should not be confused with the pressure ω. $\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$ is a function which allows for any further contributions, such as diffusive or dissipative forces and Φ is the force potential per unit mass.

The density $\tilde{\rho}(\mathbf{x}, t)$ of the GLM flow $\bar{\mathbf{u}}^L(\mathbf{x}, t)$ is a mean quantity and is defined to satisfy (4); furthermore it is connected to the actual fluid density ρ^ξ by

$$\tilde{\rho} = \rho^\xi J \; ; \; J = \det\{\delta_{ij} + \xi_{i,j}\} \qquad (5)$$

where J is the Jacobian of the mapping $\mathbf{x} \to \mathbf{x} + \xi$.
The Eulerian and Lagrangian-mean velocities are related to the Stokes drift $\mathbf{d}$, as

$$\bar{\mathbf{u}}^L = \bar{\mathbf{u}} + \mathbf{d}$$

where, for small disturbances ξ from the mean trajectories, measured by the parameter ε and an O(1) mean state,

$$d_i = \langle \xi_j u'_{i,j} \rangle + \frac{1}{2} \langle \xi_j \xi_k \rangle \, \bar{u}_{i,jk} + O(\varepsilon^3).$$

Note that $\mathbf{d} = O(\varepsilon^2)$.

THE GLM AND O(1) TURBULENT SHEAR FLOWS

We should like to apply the GLM equations to O(1) turbulent shear flows in which all mean quantities save the mean pressure are independent of the streamwise x and, to leading order, spanwise y directions. Our intent is to study instability mechanisms, if any, which operate with waves that are independent of y. We thus assume the field comprises a primary unidirectional fully developed turbulent shear flow [U(z), 0,0] with small O(δ) spanwise periodic perturbations with Eulerian velocity components (u, v, w), and small O(ε) wavelike disturbances along with associated terms of higher order in ε. Space coordinates corresponding to (u, v, w) are $(x_1, x_2, x_3) \equiv (x, y, z)$ with unit vectors ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$). Then

$$\mathbf{u}(y, z, t) = [U(z), 0, 0] + [u, v, w]$$

Crucial to the GLM formulation is the invertibility of (1) and that requires a non-zero Jacobian (5). As a practical matter, the use of Andrews & McIntyre's formulation as a computational tool in turbulent flows is limited by indeterminancy arising from the failure of this condition within a time scale comparable to the inverse of the largest Liapunov exponent in the flow. For the flows we wish to study, however, this condition is obviated (Phillips 1989).

The Lagrangian mean flow is divergence free for the above flows (Phillips 1989), but the dependent variable $\bar{u}_i^L - p_i$ suggested by (2), is not solenoidal; we thus introduce

$$\mathbf{Q} = \bar{\mathbf{u}}^L - \mathbf{p} + p_3\mathbf{k}$$

such that $\nabla \bullet \mathbf{Q} = \nabla \bullet \bar{\mathbf{u}}^L = 0$. Then since $d_3 = 0$ at O(ε), we may introduce the Lagrangian-mean stream function ψ^L, as

$$v + d_2 = \frac{\partial \psi^L}{\partial z}, \quad w + d_3 = -\frac{\partial \psi^L}{\partial y}$$

After a tedious analysis the Stokes drift and pseudomomentum become (Phillips 1989)

$$p_i = -\int_0^t A_{jk}\mathbb{R}_{kji}\,d\tau - \int_0^t U_{1,3i}\,\tau R_{31}\,d\tau - U_{1,3}\Big\{\int_0^t (t-\tau)\,[R_{13,i} + R_{31,i}]\,d\tau - \int_0^t (t-\tau)\,[\mathbb{R}_{31i} + \mathbb{R}_{13i}]\,d\tau\Big\}.$$

and

$$d_i = \int_0^t A_{jk}\mathbb{R}_{kij}\,d\tau + \frac{1}{2}U_{i,33}\int_0^t (t-\tau)\,R_{33}\,d\tau.$$

where

$$A_{jk} = \delta_{jk} + (t-s)\,U_{j,k} \quad \text{and} \quad \mathbf{r} = (s - t_o)U\,\mathbf{i}.$$

with

$$R_{kj} = \langle u'_k(\mathbf{x}_o + \mathbf{r}(s), s)\, u'_j(\mathbf{x}_o + \mathbf{r}(t), t)\rangle$$
$$= R_{kj}(U\tau, 0, 0, \tau; z)$$

and

$$\mathbb{R}_{kji} = \langle u'_{k,i}(\mathbf{x}_o + \mathbf{r}(s), s)\, u'_j(\mathbf{x}_o + \mathbf{r}(t), t)\rangle$$
$$= \langle u'_{k,i}(0, 0, 0, 0; z)\, u'_j(U\tau, 0, 0, \tau; z)\rangle$$

THE EVOLUTION EQUATIONS

In considering fully developed turbulent channel flow two obvious velocity scales enter the problem, the centreline velocity U_o and the friction velocity U_τ, and two obvious length scales, the channel half width h and the viscous length scale ν/U_τ. Together these could be used to write the equations in non-dimensional form. But Streaks are a wall layer phenomenon and appear to scale with the wall region, so an appropriate length scale is one representative of that region. We thus introduce the length scale, P say, which is characteristic of the wall region where, in wall units (Phillips, 1988)

$$P^* = PU_\tau/\nu = \int_0^\infty [\kappa^{-1}\ln(e^2 U_\tau y/\nu) - U/U_\tau]\, d(U_\tau y/\nu) \cong 68.7$$

where $\kappa (\approx 0.41)$ is von Karman's constant.
We may then interpret the characteristic wavelength of any oscillatory field that operates in the wall region as 1/P, with the frequency scale U_o/P; the amplitude of such waves, however, will be given by ν/U_τ.

On setting $\tilde{U} = U_\tau U$, where tilda means a dimensional quantity, we find

$$\tilde{\mathbf{u}} + \tilde{\mathbf{d}} - \tilde{\mathbf{p}}$$
$$= U_\tau [U + u]\,\mathbf{i} + U_o/P^{*2}\,[d_1 - p_1]\,\mathbf{i} + (U_o\nu/P)^{1/2}\,[(v + d_2 - p_2)\,\mathbf{j} + (w + d_3 - p_3)\,\mathbf{k}]$$

$$\tilde{\mathbf{x}} = P\,[y\,\mathbf{j} + z\,\mathbf{k}], \quad \tilde{t} = \frac{P}{U_o}\left(\frac{U_o P}{\nu}\right)^{1/2} t$$

$$\tilde{\psi} = P\left(\frac{U_o\nu}{P}\right)^{1/2}\psi \quad \tilde{\zeta} = \left(\frac{U_o\nu}{P}\right)^{1/2}\frac{\zeta}{P}$$

The Langmuir number La (Leibovich, 1977) is the only non-dimensional parameter to appear with this scaling, where

$$La = (PU_o/\nu)^{-1/2} = P^{*-1/2}\,(C_f/2)^{1/4}$$

Observe that because the skin friction coefficient C_f varies only logarithmically with momentum thickness Reynolds number R_θ (Phillips 1988), La is close to constant over the experimental range. In particular at $R_\theta = 500$, La = 0.0269 while at $R_\theta = 10^4$, La = 0.0231.
In dimensionless form, the evolution equations become

$$\zeta = -\nabla^2\psi^L, \qquad (6)$$

$$\left(\frac{\partial}{\partial t} - La\nabla^2\right)u = J(\psi^L, u) + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial z}\{U + (P^*La)^{-2}\,(d_1 - p_1)\} \qquad (7)$$

and

$$\left(\frac{\partial}{\partial t} - La\nabla^2\right)\zeta = J(\psi^L, \zeta) - \frac{\partial p_1}{\partial z}\frac{\partial u}{\partial y} \qquad (8)$$

Note that over the experimental range of Reynolds numbers $P^*La = O(1)$.

Non-decaying solutions to (7) and (8) are admissible only with, but are not guaranteed by, non-zero coupling terms. For non-zero coupling terms a fluctuating field, whose rectified effect manifests as the pseudo-momentum, must occur simultaneously with a weak spanwise distortion of the mean flow. A third ingredient, not necessary for coupling but important for instability, is the presence of shear in the mean flow: attest the fact that Streaks occur only in the wall region of a turbulent boundary layer, where the shear is highest. Shear is also necessary for the Craik-Leibovich mechanism to operate. Note that (7) and (8) recover the governing equations for Langmuir circulation in which, for $O(\varepsilon^2)$ mean flows, $p_1 = d_1$ (Leibovich 1977).

For boundary conditions we shall assume no slip at the wall, that all Eulerian velocities are symmetric on the channel centre line and that the flow is spanwise periodic, then

$u = \zeta = \psi = 0$ on $z = 0$;

$du/dz = \zeta = \psi = 0$ on $z = h/P$; and

$u(0, z) = u(L, z), \zeta(0, z) = \zeta(L, z), \psi(0, z) = \psi(L, z)$.

The period L (= $\tilde{L}/P = L^*/P^*$), however, is unknown and may be determined in from a linear stability analysis by identifying L as the most unstable lateral mode.

STABILITY ANALYSIS

Spanwise periodic perturbations must, on assuming normal modes, take the form

$$[u, \psi] = \delta \mathrm{Re}\, \{e^{\sigma t} e^{ily}\ [B(z), -\varepsilon/il\ W(z)]\}$$

We note that the perturbation velocity components in the y- and z-directions are weaker than the downstream component by a factor ε; this is perhaps why cross-stream velocity components do not register in conditionally averaged hot-wire measurements! The resulting equations are then

$$[D^2 - l^2 - \tilde{q}]\, B = (P^*La^{-1})\, W\, dU/dz \tag{9}$$

$$[D^2 - l^2 - \tilde{q}]\, [D^2 - l^2]\, W = -(P^*La)^{-1}\, l^2\, \{B\, dT_1/dz - \tilde{T}_1 dU/dz\}\ , \tag{10}$$

where l is the spanwise wavenumber and $\tilde{q}(l)$ is the growth or decay rate to be found.

Equations (9) and (10) must satisfy the boundary conditions

$$B = D^2W = W = 0 \text{ on } z = 0,\ dB/dz = D^2W = W = 0 \text{ on } z = h/P. \tag{11}$$

The right-hand side of (10) depends upon the distortion of the wave field through $\tilde{T}_1$, but, as Craik (1982) points out, the GLM equations provide no direct means of evaluating $\tilde{T}_1$ and a separate examination of the wave field is necessary. For our present purposes, however, we shall set $\tilde{T}_1 = 0$.

The eigenvalue problem defined by (9), (10) and (11) bears similarity to a number of classical stability problems (see Drazin & Reid 1981), in particular Taylor-Görtler, but it is *not* Taylor-Görtler.

RESULTS AND DISCUSSION

Calculations were done at the Reynolds number of Kreplin & Eckelmann's (1979) experiments, viz 385 (based on channel half width). The Stokes drift and pseudomomentum were evaluated using an empirical expression for the space-time correlations based upon one given by Favre (1965); the mean velocity and turbulence intensity profiles were those of Phillips (1987) and Phillips & Ratnanather (1989). With this input the linear stability analysis yielded a value of L = 120 wall units for the most unstable spanwise mode. Since the linear analysis is the initial value problem to the fully non-linear problem, the solution to it was used as input to the non-linear case. The numerical details are given in Phillips (1989).

The instantaneous cross-sectional streamlines of the vortices do not change greatly with time, see figure 2, but their strength increases. Initially the increase is gradual and

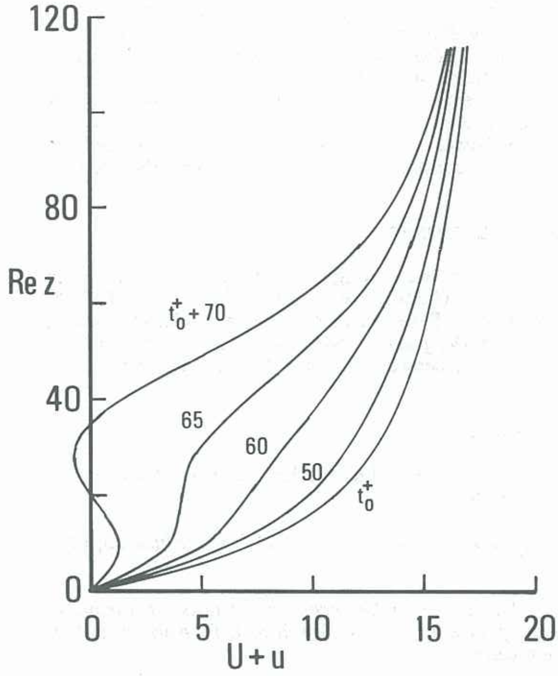

Figure 1. Evolution of streamwise velocity profile on symmetry axis between two vortices.

would appear to be due to enhancement by the mechanism generating the vortices, rather than the instability arising from the inflexion point they induce in the streamwise velocity profile. But eventually that instability dominates and the vortices undergo an exponential growth. During this period the streamwise velocity perturbation grows to O(1), see figure 1, and is coupled with an explosive upthrust of fluid from the wall. The latter phenomenon is observed experimentally and is termed Ejection. Profiles of constant velocity one wall unit above the wall are given in figure 3. These are initially symmetrical but do not remain so; eventually stagnant regions of spanwise spacing L, form. It is evident that any dye injected close to the wall would accumulate in these regions, appearing to an observer above as Streaks. Moreover, some of the dye would slowly rotate with the vortices - remember that their rotational velocities are two orders of magnitude lower than the streamwise

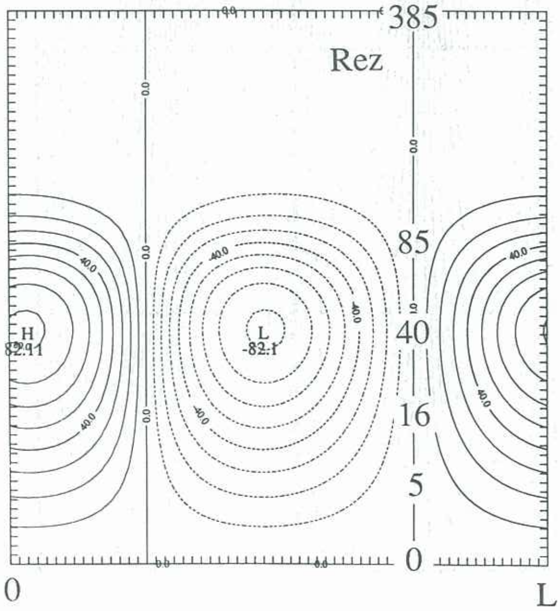

Figure 2. Instantaneous cross-stream strealines at $t_0^+ = 90$.

velocity perturbation. These helices, viewed from above, appear as oscillations, or what Kline et al (1967) termed Breakdown. Throughout the entire process the vortices rise gradually from the wall. It appears that their height following Ejection is such that the dynamic instability no longer dominates. Whether this is because they are now in a region of lower shear or for other reasons is unclear. Following Ejection, new vortices form and the process repeats.

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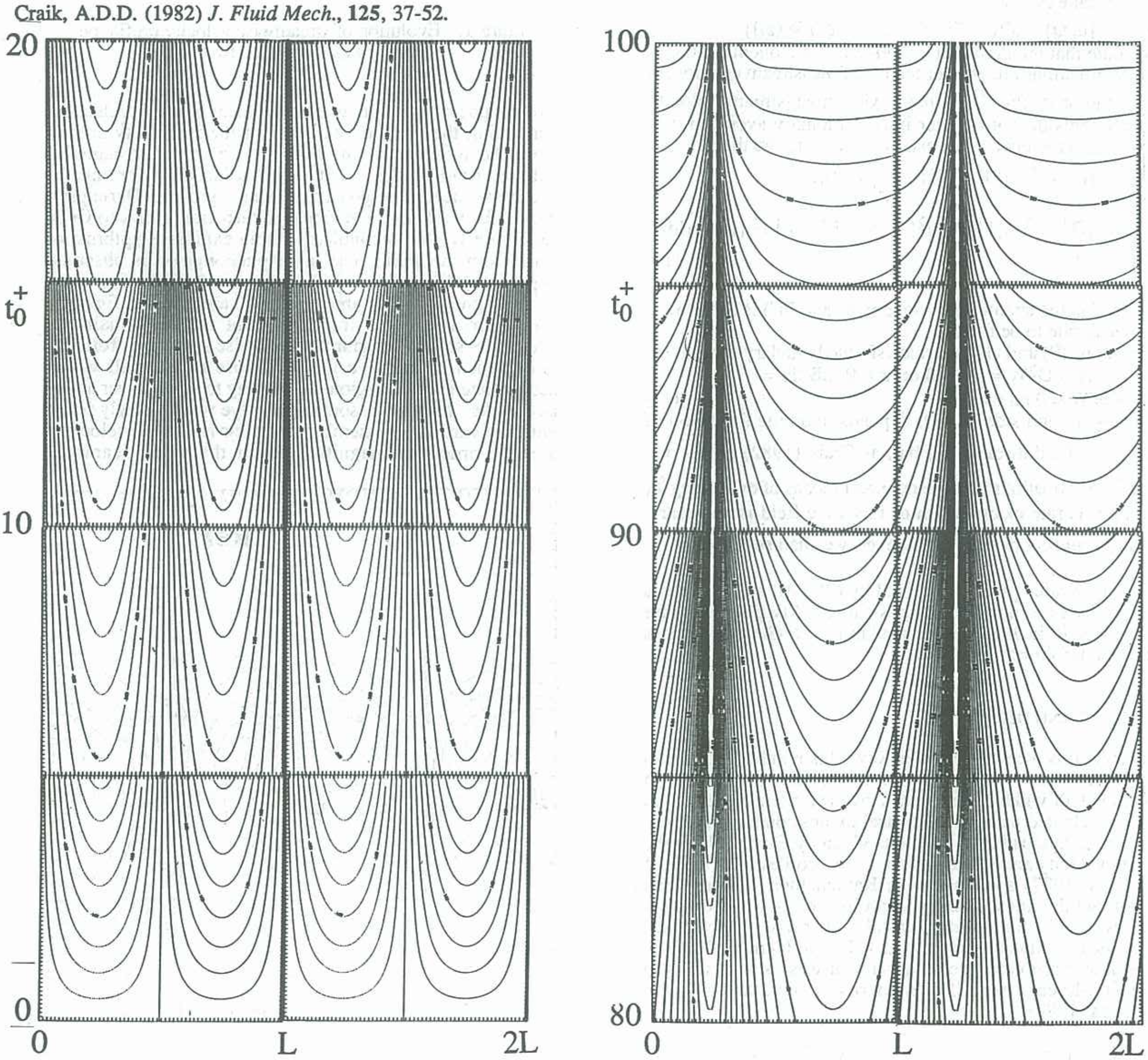

Figure 3. Evolution of contours of constant velocity in a plane one wall unit above the boundary.