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BUILDING BRIDGES BETWEEN DYNAMICAL CHAOS AND OPEN-FLOW TURBULENCE

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1 INTRODUCTION

Recent developments in the theory of dynamical systems exhibiting chaotic behaviour (summarised e.g. by Guckenheimer et al 1983, Wiggins 1988) have raised the question whether turbulence in fluid flows could be understood as dynamical chaos. Several proposals on possible routes to chaos have been made, and these have found some support from observations in bounded flows such as convection in a box (Gollub et al 1980). However, the existence of any connection between the dynamical chaos exhibited by such systems and turbulence in open flows like boundary layers and wakes has frequently been questioned. It is generally felt that the chaotic phenomena observed in low-dimensional nonlinear systems may be relevant only to 'weak' turbulence, i.e. to stages preceding onset of fully turbulent behaviour (Hao 1984, Morkovin et al 1987).

The questions mentioned above crystallise into three basic issues (Narasimha 1987).

(i) Chaotic dynamical systems do not appear to exhibit a strong cascade process of the kind generally considered an essential feature of turbulence (Batchelor 1953). *Figure 1* compares the spectrum of the Lorenz system (Lorenz 1963) with that of a typical turbulent flow, showing how the energy in the former falls off steeply with increasing frequency, and suggesting that the chaos of such low-dimensional systems is basically 'slow' (as one may expect from the commonly encountered period-doubling route to chaos).

(ii) In open flows, especially boundary layers and ducts, the critical value of the control parameter (e.g. Reynolds number) at onset of turbulence is *not* unique, but depends strongly on the disturbance environment. Indeed, an analysis of experimental data on boundary layers that allows for the presence of residual non-turbulent disturbances in the facilities used for testing strongly suggests that the critical Reynolds number is inversely proportional to the disturbance intensity (Govindarajan et al 1989; see *Figure* 2).

(iii) Flow turbulence persists for values of the control parameter beyond the critical, but low-dimensional dynamical systems exhibit order and chaos in alternating windows; in the Lorenz system the solution at sufficiently high values of the Rayleigh number is just a limit cycle (Sparrow 1982).

The question that arises is whether these criticisms apply only to dynamical systems considered to date. In the present lecture I shall describe a model (Narasimha et al 1988) whose behaviour appears to answer the above criticisms and mimic certain generic features of open-flow turbulence. The model is only "impressionistic" (in the sense used by Narasimha 1989), and is intended to offer insight rather than quantitative predictions for any particular flow. Detailed numerical and analytical studies of the model are available in Bhat et al (1989) and Bhat et al (1989), which will be referred to as I and II below.

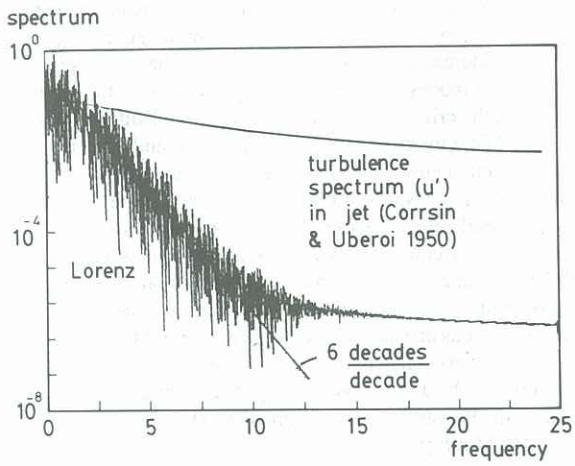
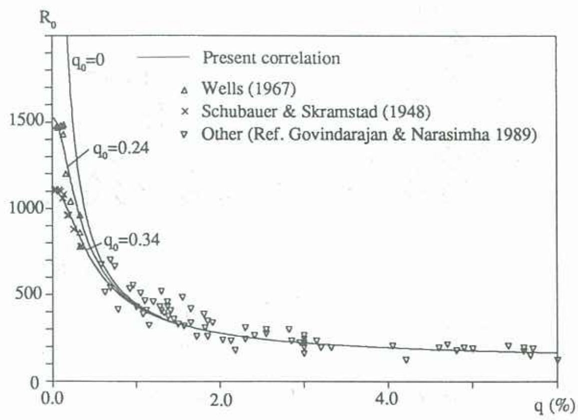


Figure 1: Comparison of a typical spectrum of the solution of the Lorenz model with that of the streamwise fluctuating velocity component in a turbulent jet (Corrsin & Uberoi 1950), normalised to agree at the lowest frequencies. The flat spectrum of the Lorenz model beyond a frequency of about 10 is a numerical artifact.



Transition onset Reynolds number for zero pressure gradient flow

Figure 2: Dependence of transition Reynolds number R_t in a boundary layer on free-stream turbulence q; at low values of q transition is driven by facility-specific non-turbulent disturbances, parameterised by the variable q_0. The full lines show a correlation with $(q^2 + q^2)$ proposed by Govindarajan & Narasimha (1989).

A different kind of attempt to link relatively low-dimensional chaotic dynamics and an open flow system has been made recently by Aubry et al (1988). They model the behaviour of streamwise vortex rolls near the wall in a boundary layer and show that it is possible to capture the ejections and bursting events observed experimentally. Their study, utilising modes derived from observation, confines itself by design to a specific flow, and in particular is not concerned with transition. The present model, on the other hand, is intended to look at open flows in general, including transition mechanisms.

Some very interesting experimental work has been reported on wakes (Sreenivasan 1985, Van Atta et al 1987, Olinger et al 1988); the central issue here appears to be the necessity or otherwise for the presence of forcing of some kind to induce chaos.

2 THE MODEL

The basic philosophy in constructing the present model is to incorporate in it those physical factors that appear essential to turbulent behaviour, at the same time retaining simplicity to enable detailed analysis. The conventional method of using a truncated Galerkin approximation to get a set of ordinary differential equations from the governing partial differential equations will in general need a large number of modes, and hence also equations, even to *attempt* meeting the criticisms listed in Section 1. This difficulty is avoided here by treating turbulent flow as mainly the outcome of interaction between motions at two widely different scales, somewhat in the spirit of Liepmann's (1961) turbular fluid. More specifically the spectral or wave number space is considered to consist of two broad regions, one where non-linearity and external disturbances play the major role (representing the so-called large eddy motion) and the other where viscous dissipation is dominant, representing the small or Kolmogorov scale motion (see *Figure 3*). These two scales are coupled by a non-linear energy transfer mechanism, often called the Richardson cascade process, schematically illustrated in *Figure 4*.

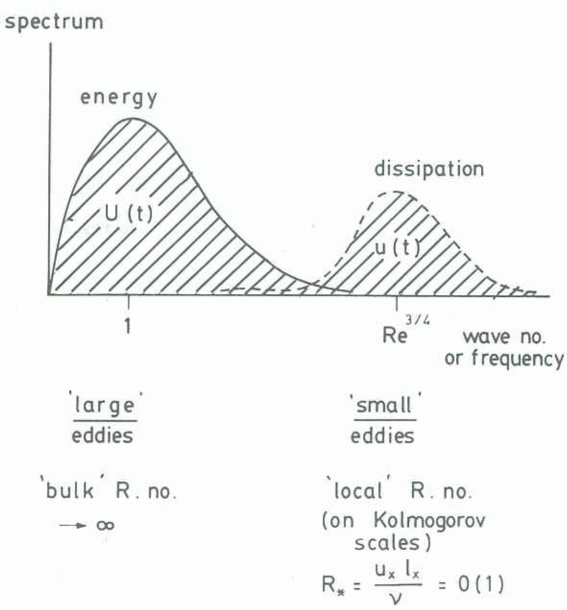


Figure 3: A sketch of the spectrum of energy and dissipation in turbulent flow, indicating their relation to the variables of the present model.

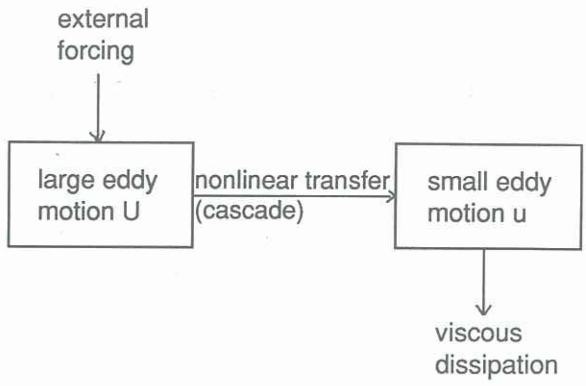


Figure 4: Schematic diagram to illustrate the cascade process in turbulent flow.

The model has two dependent variables U, u, both considered functions only of time t, and a specified external forcing q(t). U and u may be thought of as representing the amplitudes characteristic of large and small eddies respectively, the actual velocity being a combination like:

$$a\, U(t) \exp i(K_1.x - \omega_1 t) + b\, u(t) \exp [i(K_2.x - \omega_2 t + \phi)] \quad (1)$$

where K_1, ω_1, and K_2, ω_2 are characteristic large and small eddy wave numbers and frequencies and a and b are weighting functions that are a measure of the intensity of as well as the bandwidth covered by the respective motions in wave-number space. For the sake of simplicity we shall assume that the eddies have a common celerity $\omega_1/|K_1| = \omega_2/|K_2|$, and that time is measured in a coordinate frame moving with the fluid at this speed, invoking a form of Taylor's hypothesis. Note the significant departure from earlier studies, typified by the Lorenz system, where only the lowest few modes are selected for describing the dynamics. If u(t) is seen as representing Kolmogorov-scale motions, we should expect the weighting factor b to be a function of the Reynolds number characterising the large-eddy motion (Batchelor 1953). An alternative interpretation that is helpful in motivating the model is inspired by experimental observations in boundary layer transition (see e.g. the recent reviews by Stuart 1986 and Herbert 1988), where it is found that as the amplitude of oscillation in the primary Tollmien-Schlichting instability increases, an intense small-scale high-shear layer develops; this secondary instability rapidly leads to the growth of fine scale motion.

The importance of including a forcing term q(t) is evident not only from classical work (Schubauer et al 1948) on boundary layers, in particular as analysed by Govindarajan et al (1989), but also from the recent work of Gaster (unpublished). It is our premise here that in boundary layers certainly, and possibly in all open flows, there is no transition if there is no forcing (ignoring microscopic triggering such as that due to Brownian fluctuations). (As an aside, therefore, one way of keeping a flow laminar would be to provide it with a quiet environmental sheath, itself maintained possibly by active control operating on the receptive frequency bands.)

The following physical considerations were used to suggest possible forms for the model:

(i) The large scale variable U must be governed by a control parameter that plays the role of Reynolds number (denoted by R say), whose variation changes overall system behaviour.

(ii) There must be a critical value R_c of this parameter such that for $R < R_c$ the motion is stable, and for $R > R_c$ the system exhibits linear instability.

(iii) For $R > R_c$ the growth of U because of linear instability will eventually be checked by non-linearity and saturate at some finite maximum that depends on R.

(iv) The value of R at which onset of chaos will occur, say R_t, will in general be higher than R_c and depend on the forcing q(t) (which may be deterministic).

(v) The small-scale motion u will gain energy from the large-scale motion U by non-linear interaction.

(vi) Energy at the small scales is lost due to direct viscous action.

Several non-linear models may be written down that are generally consistent with the features listed above. We shall discuss here the model called System 2 in Narasimha et al (1988), as it was found to be the simplest that has many desirable and interesting properties. The model is described by the equations:

$$dU/dt = U(1 - \upsilon - U^2) - K\,u\,|u| + q(t) \qquad (2a)$$

$$du/dt = k\,U(|u| + \sigma) - \hat{\upsilon}u, \qquad (2b)$$

with the forcing taken in general to be the sum of periodic and stochastic components:

$$q(t) = \overline{q}\cos\omega t + q_n\,\xi(t). \qquad (2c)$$

It is necessary to discuss briefly the different terms in equations (2). First of all, the linear and cubic terms in U in (2a) are prompted by the non-linear stability theory of Stuart (1960) and Watson (1960) for plane Poiseuille flow. The parameter υ represents the effect of viscosity on the large-scale motion. With an expression like (1) for the velocity, the viscous term in the Navier-Stokes equations will be proportional to the actual viscosity of the fluid times K_1^2 and K_2^2: the υ and $\hat{\upsilon}$ of (2) are to be thought of as these *products*. The limit υ --> 0 is like the Reynolds number tending to infinity in a flow, and (2a) reflects the fact that at high Reynolds numbers non-linear interaction dominates over viscous effects on the large scales (cf. the principle of high Reynolds number similarity (Townsend 1976), according to which the viscosity is asymptotically irrelevant in large-eddy motion as the flow Reynolds number tends to infinity). Clearly υ plays the role of control parameter here and may be thought of as proportional to the inverse of a Reynolds number based on large-eddy scales. The coefficients K and k govern the non-linear interaction between large and small eddy motion, and represent the net effect of the cascade which actually takes place through intermediate scales. The small eddy variable u grows because of interaction with U and is assumed to act roughly like a Reynolds stress on U as given by the second term in (2a). The parameter σ, assumed small, is added to ensure that u is always excited in the presence of U. We have found that it plays no great role in the model, but in its absence u = 0 is always a solution. In practice, numerical noise ensures that u does not remain zero even if $\sigma = 0$, but we have preferred to put in explicitly an 'agitation' from the large eddies instead of leaving it to numerical 'stirring'. At the small scales, viscous dissipation is represented by the term containing $\hat{\upsilon}$, which is assumed to be always significant; so $\overline{\upsilon}$ does *not* vanish with υ (K_2 adjusting itself suitably in the limit), and this is a crucial feature of the present approach.

The forcing (2c) permits us to mimic experimental studies undertaken to elucidate the mechanisms underlying transition, where free-stream turbulence and other stochastic disturbances have often been greatly reduced and artificial periodic forcing introduced (e.g. Schubauer et al 1948, Nishioka et al 1975, Gaster 1984).

The use of the absolute value sign in (2) introduces a symmetry in the (U,u) phase plane and actually results in a simpler system (the transformation U --> -U, u --> -u leaves equations (2a,b) invariant). The chief physical implication is that in spite of the cascade process, energy transfer between large and small eddies is not always in the same direction - a fact that is well recognised (Batchelor 1953).

It is seen that, apart from the forcing, there are five unspecified parameters in the model. Out of these σ is assumed small and has been assigned a value of 0.05 in all the investigations reported here. The choice of values for the other parameters is guided by the solutions of the unforced system, which we now briefly describe.

3 THE STRUCTURE OF THE SYSTEM

It can be shown (I) that the unforced system has, apart from the origin, two other fixed points located symmetrically in the phase plane. Their position depends chiefly on υ and the ratio $\delta = \hat{\upsilon}/k$, which represents in some sense the ratio of viscous dissipation to non-linear energy transfer. As there is a balance between the two in turbulent flow, the viscous scales adjusting themselves to dissipate the energy that cascades down from the large eddies, we shall require δ to be 0(1).

The behaviour of the system does not depend strongly on σ, K or k, but does depend critically on υ and $\hat{\upsilon}$. To illustrate the nature of the attractors in the system, we show this dependence in *Figure 5a*, for $\sigma = 0.06$, K = k-3.3. It is seen that at $\upsilon = 1$ there is a pitchfork bifurcation, giving rise to two additional fixed points for $\upsilon < 1$. They are situated respectively in the first and third quadrants of the phase plane, their precise location depending mainly on υ and δ; as an example, their variation with υ for $\delta = 0.55633$ is shown in *Figure 5b*. It is seen that as υ --> 0 the position of the large-eddy fixed point U_* is insensitive to υ which is consistent with the principle of high Reynolds number similarity already cited.

A homoclinic orbit plays a special role in dynamical systems because it can break even under very small perturbations giving rise to chaos. Since it should take little external forcing to trigger turbulence at high Reynolds numbers, we should want the present model to become very sensitive to external forcing as σ --> 0. The value assigned for δ (and $\overline{\upsilon}$) above is precisely that required to ensure this, i.e. to give a homoclinic orbit at $\upsilon = 0$.

The fixed points of the unforced system may be thought of as representing states of motion preferred in some way; the presence of two fixed points apart from the origin indicates the existence of two such preferred states at $\upsilon < 1$. There are many flows that exhibit such behaviour. For example, the wake behind a body (especially one that is blunt) contains vortices of opposite sign in a Karman vortex street when the Reynolds number is not too low; such vortices are known to persist in some form at very high Reynolds numbers as well (Roshko 1961, Cantwell et al 1983). Each fixed point may therefore be seen as the analogue of a vortex of one sign. As a second example, we may cite the (turbulent) boundary layer in which the flow may largely be described as the

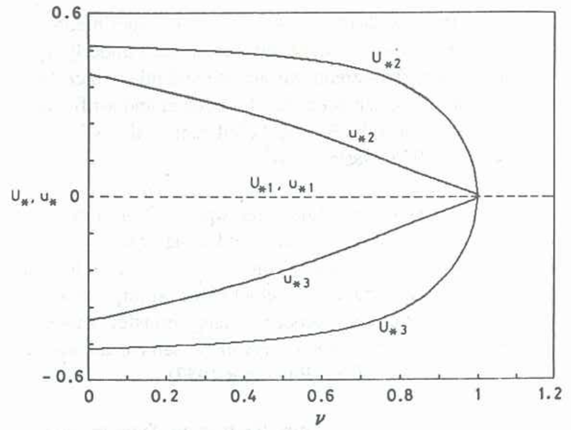


Figure 5a: Dependence of the fixed points U_* and u_* on υ for K = 2.3 and δ = 0.55633. There are three fixed points for υ<1; the non-zero fixed points 2 and 3 are at the same distance from the origin, but located in opposite quadrants.

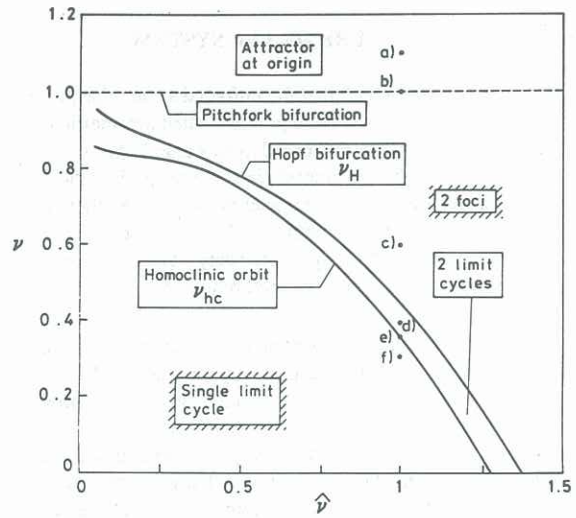


Figure 5b: Bifurcations and attractors in the unforced system at K=k=2.3.

succession of two basic patterns of motion called respectively ''sweeps'' and ''ejections'' (Kline et al 1967, Corino et al 1969). It is therefore useful to think of the two non-zero fixed points in the present dynamical system as indicating the possibility of two characteristic patterns of motion that constitute (latent) coherent structures.

4 RESPONSE TO PERIODIC FORCING

With forcing, even when it is purely sinusoidal (q_n = 0 in (2c)), it can be shown that the deterministic system (2) does indeed possess chaotic solutions under certain conditions (I).

Here by chaos we shall mean the irregular behaviour of the system as a result of repeated stretching and folding of volumes in phase space (Wiggins 1988), with the associated property of sensitive dependence on initial conditions. Inferences about the existence of chaos may be made from numerical studies of Poincare maps. These are obtained by sampling the solution at time intervals equal to the period of forcing; i.e. successive points are iterates or images of their predecessors one period earlier. In spirit this procedure bears some resemblance to the technique of conditional sampling used in the search for persistent ''coherent'' structures in turbulent flows.

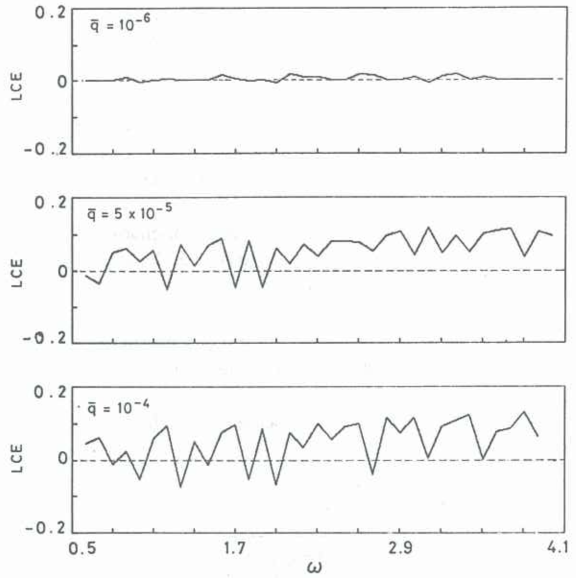


Figure 6: Dependence of the Lyapunov characteristic exponent on ω and q at υ = 0. Even at q = 10^{-6}, the exponents are positive at some frequencies.

Indeed, using a technique due to Melnikov (1963), it can be demonstrated analytically (I) that for appropriate values of k and $\hat{\upsilon}$ (including those chosen here), the system possesses homoclinic tangles, and hence exhibits chaos, *for all sufficiently small values of* υ under forcing: a property that is very important in the light of the criticisms of Section 1.

From a fluid-dynamical viewpoint, the important problem is to identify the range of υ and $\bar{q}$ where the system possesses chaotic solutions. For this purpose, the most convenient parameter to study is the Lyapunov Characteristic Exponent (LCE), which is defined as

$$\lambda = \lim_{\substack{t \to \infty \\ d(O) \to 0}} ln(d(t)/d(O))$$

where d(t) denotes the separation between two orbits at time t. In general, a system with n degrees of freedom has n characteristic exponents, and whenever at least one of them is positive the system goes chaotic.

In the present model the LCE depends on υ, $\bar{q}$ and ω; as an example, *Figure 6* shows the values as a function of ω for three values of $\bar{q}$ at υ = 0. In this figure, ω is varied in steps of 0.1 over the range of 0.6-4.0 and successive points are joined by a straight line. It is seen that for a given $\bar{q}$, the LCE is positive in certain frequency bands, i.e. the model is more ''receptive'' at certain frequencies (using the word in the sense of Morkovin 1969). Further it is seen that even for $\bar{q} = 10^{-6}$ there are frequency bands where the LCE is positive, suggesting that the system is chaotic even at the smallest forcing (as υ --> 0), although there can be no chaos when the forcing vanishes. Similar features are observed at higher values of υ also, but as υ increases the ''receptive'' ranges of ω narrow down, and a higher forcing amplitude is required to induce chaos. The latter point is brought out in *Figure 7* where the boundary separating LCE > 0.01 from smaller and negative values is shown (due to the difficulty of accurately estimating the LCE, we have adopted the operational definition that the system is chaotic whenever LCE exceeds 0.01). It is seen that the boundary in Figure 7 turns back on itself, the region between the upper and lower curves covering the chaotic regime for the system. The lower curve depicts the

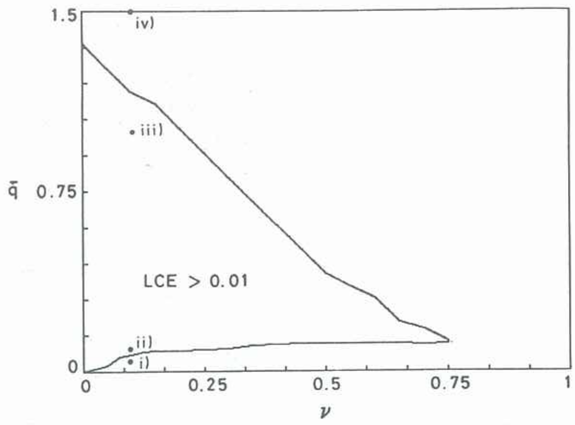


Figure 7: Boundary separating chaotic from non-chaotic regimes in the model with periodic forcing. Note that for υ>1 the system is stable, and that for υ>0.76 chaotic behaviour is not possible. Across the lower boundary there is transition to chaos as q increases at fixed υ, or as υ decreases at fixed q. Across the upper boundary, high forcing induces periodic behaviour, as in relaminarisation by domination.

minimum forcing required to induce chaos, and may be thought of as mimicing the known dependence of transition Reynolds number R_t on the external disturbance level (Figure 2). Above the upper curve the system exhibits forced periodic oscillations, much as the wake behind the cylinder does if the cylinder is itself oscillating with large amplitudes (Williamson et al 1988); the system in this case may be regarded as exhibiting relaminarisation by the mechanism identified as "domination" by Narasimha et al (1979). It is interesting to note that, as υ increases, the lower and upper curves approach each other and there is a value of (≈0.76) above which no chaos is possible according to the present criterion; this number may be thought of as corresponding to the lowest possible transitional Reynolds number $R_{t\,min}$ in a flow under sinusoidal forcing.

A detailed study of the routes to chaos in the model reveals that there are mainly two types of transition scenario (II). The first is seen when the system is forced near its natural frequency; the periodic cycle, when it becomes unstable, undergoes a cascade of periodic-doubling bifurcations, following closely a Feigenbaum (1978) scenario. The second, observed when the natural and forcing frequencies are widely separated, leads to the onset of chaos directly from a frequency-locked state (usually known as the quasi-periodicity route) and resembles the transition seen in a circle map (Arnold 1965, Thompson et al 1987).

5 RESPONSE WITH A STOCHASTIC COMPONENT IN THE FORCING

In practice, some noise is always present in any experiment, especially in fluid flows, so there is practical interest in understanding how the system behaviour is modified when a stochastic element is present in the forcing. We assume that the forcing is a combination of a periodic and a stochastic term, as in controlled experiments subjecting the flow to a periodic excitation which nevertheless is contaminated by a small amount of noise. Two kinds of stochastic forcing, respectively a Gaussian pink noise (referred to as N1) and an amplitude-limited white noise (N2) have been employed for the purpose. The method of Poincare sections is of little use in the present situation. Our main tool has to be spectral analysis, which has several advantages. In particular, the power spectrum clearly brings out the relative importance

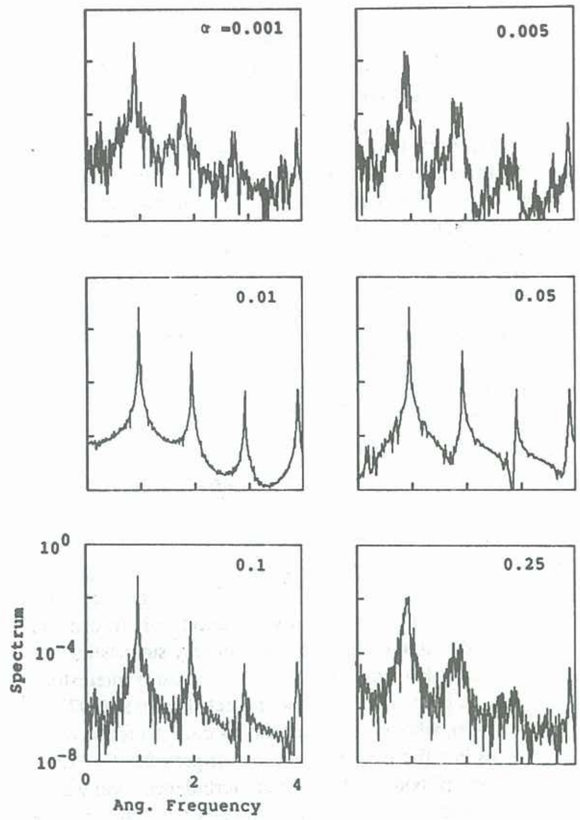


Figure 8a: Changes in the spectrum of U with forcing at ω = 1, υ = 0.1 and 5% noise (q_n/q = 0.05).

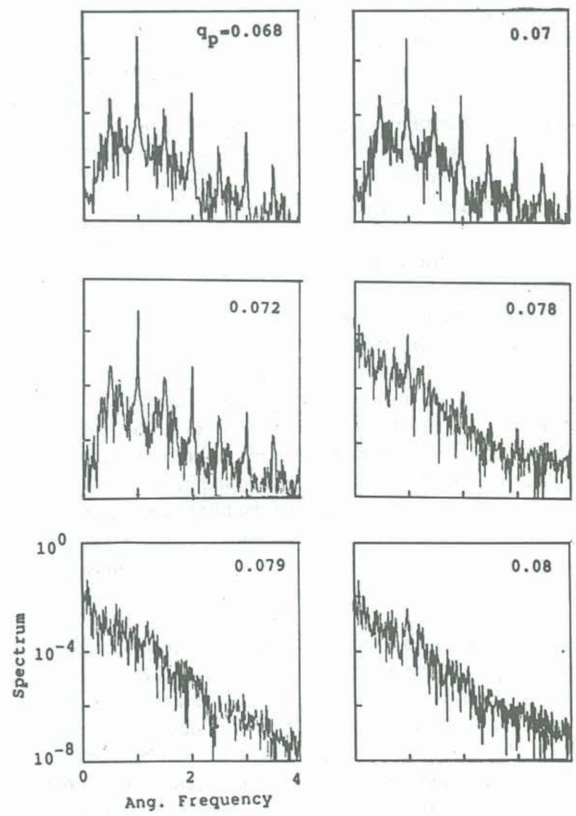


Figure 8b: Influence of increasing levels of N1 when the deterministic system is frequency locked. Other parameters are υ = 0.025, q_p = 0.05 and ω = 3.9. The wide-band spectrum at α = 0 is due to the transient chaos. Transients vanish faster in the presence of a moderate amount of noise, e.g. at α = 0.01 and 0.05.

of different modes and can give much more insight into the changes in the solution as the control parameter varies. Typical results are shown in *Figure 8*. Analysis with N1 and N2 leads to the following picture.

In general the presence of a stochastic term in the forcing influences the route to chaos considerably. When the system is in the period-doubling regime, the number of bifurcations that can be identified goes down as the noise level increases. This is in agreement with the results of Crutchfield et al (1980). It is interesting to note that when the steady behaviour is a frequency-locked state, the system is less sensitive to the presence of noise. In fact a moderate amount of noise appears to make the system more strongly periodic. Also the spectral distribution of noise influences the transition process considerably.

6 COMPARISONS WITH OBSERVATION

Is there any correlation between what has been observed in this model and in real open flows? Making this comparison is still rather difficult chiefly because there is no experimental study where the flow is periodically forced and the spectral distribution of other (presumably stochastic) disturbances is also monitored. Nevertheless, it is interesting to examine in particular the work of Kachanov et al (1977, 1984a, 1984b), who have studied the spectral evolution of the fluctuations in a flat-plate boundary layer periodically excited by a vibrating ribbon; the free-stream turbulence level was maintained below 0.04% of the mean velocity. Representative results from these experiments are shown in *Figure 9*. Kachanov et al (1984b) and Gaster (unpublished) report that the forcing amplitude is an important factor in determining the type of transition. For relatively small amplitudes (typically less than 1% of the free-stream velocity), one sub-harmonic and several harmonics of the forcing frequency were observed before the appearance of a wide-band spectrum; the low frequency end of the spectrum (i.e. low relative to the peak) developed more rapidly than that at high frequencies (*Figure 9a*). On the other hand, for large forcing amplitudes, the high frequency end underwent rapid changes and the low frequency end remained relatively unchanged (*Figure 9b*).

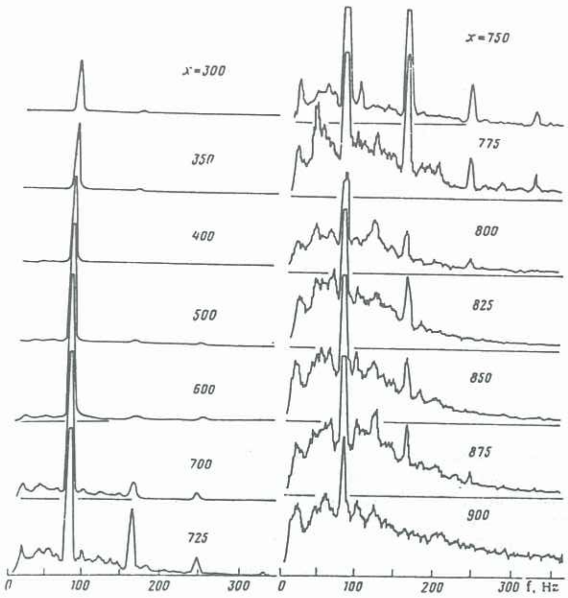


Figure 9a: Spectral evolution of the fluctuations in a boundary layer periodically forced at 81.4 Hz by a vibrating ribbon at a distance of 190 mm from the leading edge; measurements at the height of 1.4 mm above the surface. The free-stream velocity U_∞ is 9.18 m/s and free-stream turbulence level less than 0.04% of U_∞ (Kachanov et al 1977). Note the appearance of a sub-harmonic (frequency about 40 Hz) at x = 700 mm (some other frequencies can also be seen but are not so easily identified), and at least three super-harmonics at and beyond x = 725 mm. Beyond x = 800 mm the low-frequency spectrum changes little, but the high frequency end suggests that continuous development is taking place even at 900 mm.

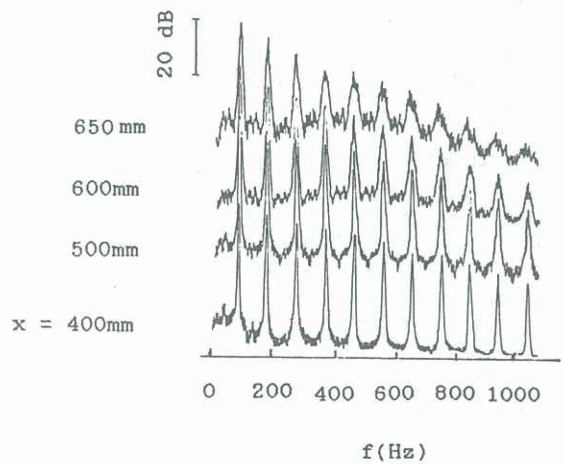


Figure 9b: Spectral evolution of the fluctuations in a boundary layer periodically excited at a distance of 250 mm from the leading edge at 96.4 Hz; measurements at a height of 4.5 mm from the surface. Free-stream velocity is 9.18 m/s and turbulence level less than 0.02% (Kachanov et al 1984a). Note how the low frequency end of the spectrum shows little change over the distance covered by the measurements, whereas the high frequency end shows continuous evolution.

These observations show some qualitative similarity with the model. For example, both Figures 8a and 9a show a rapid rise in the energy content at low frequencies. In Figure 8a it may be demonstrated (I) that the system has a tendency to undergo period-doubling bifurcations (masked to some extent if there is a stochastic component in the forcing), and this leads to the rapid filling of the spectrum on the low frequency side. Though extending this argument to Figure 9a is not strictly justified, it is tempting to hypothesise that the latter reveals a tendency for period-doubling bifurcations and that something similar to the Feigenbaum route (rendered fuzzy because of free-stream turbulence) is being followed in the boundary layer. Similar reasoning may be used while comparing Figures 8b and 9b in both of which the high frequency side appears to grow more rapidly than the low frequency end. It can be shown (II) that Figure 8b describes a situation involving frequency locking in the model and that chaos in this case was a result of loss of frequency locking (along what may be called a modified Ruelle-Takens scenario). It is therefore plausible that the transition seen in Figure 9b is following this modified Ruelle-Takens scenario, again rendered fuzzy by free-stream turbulence.

The model results show that the exciting frequency is an important parameter. It is not possible to say whether this is so in the real flows too, as in the experiments analysed above the frequency was not varied over a wide range. It would however be not surprising if the frequency played a less dominant role in a boundary layer, as the Tollmien-Schlichting instability covers a continuous band of frequencies. Both model and experiment (Meier et al 1987) show that the spectrum of the stochastic component is important.

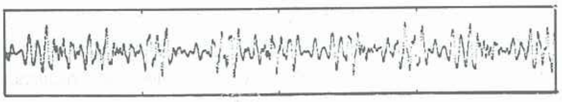


Figure 10: Example of velocity trace using the linear combination (1).

Finally *Figure 10* shows a typical velocity signal from the present model combined as in (I), and it will be seen to be not unlike measured turbulent fluctuations.

7 CONCLUSIONS

We have presented a simple dynamical model with many characteristics similar to those of open fluid flows. The cascade process is built into the model by postulating a transfer from large eddy motion (providing one dependent variable) to small eddy motion (second variable). The model contains a parameter υ which is the analogue of the inverse Reynolds number, so normalised that linear instability sets in at $\upsilon = 1$ (i.e. $\upsilon \equiv R_c/R$ in the notation of Section 2). All unforced solutions of the model are nonchaotic. With periodic forcing the model is ''receptive'' to certain frequencies or frequency bands, and can be shown to exhibit chaos by demonstrating the existence of homoclinic tangles, horseshoes and positive Lyapunov characteristic exponents. The value of υ at onset of chaos, υ_t, depends in general on both the amplitude $\bar{q}$ and frequency ω of the forcing. However if the frequency is treated as a hidden variable, i.e. we look for the lowest forcing amplitude that induces chaos at some frequency, a boundary that encloses the chaotic regime can be found in the $(\bar{q},\upsilon)$ plane. Along the lower boundary of this regime the forcing level required to trigger chaos goes to zero as $\upsilon \to 0$, though in its absence there is no chaos. The system therefore has the property that chaos persists in the high Reynolds limit. On the other hand there is no chaos for $\upsilon > 0.76$ for any frequency of the sinusoidal forcing considered; this value of υ corresponds to the critical Reynolds number below which turbulent flow is not possible under periodic forcing. At extremely high value of the forcing (above the upper boundary of the chaotic regime in the $(\bar{q}, \upsilon)$ plane), the system is in forced oscillation, as may be expected if the flow relaminarises by the mechanism of domination.

The route to chaos with periodic forcing is not unlike those observed in closed flow systems. With an additional stochastic component in the forcing, the transition process is qualitatively similar to that observed in experimental studies of a periodically excited boundary layer, and is found to depend on both the intensity and spectrum of the forcing.

It is hoped that the present work encourages the view that flow turbulence (including that in open systems) could be understood as one (albeit complex) instance of dynamical chaos.

It is unlikely that all open flows can be put into the same basket: a distinction will probably have to be made between convectively and absolutely unstable flows. But the present work does prompt a conjecture on wakes. Wakes have low instability-critical Reynolds numbers, and, when unstable, high amplification rates. Going by the evidence on the somewhat similar situation encountered in boundary layers subjected to adverse pressure gradients, we may expect less sensitivity to environmental disturbances, at least at the levels usually prevalent in wind tunnels (even the quieter ones). However, in the present view, a much greater sensitivity should be observed when the disturbance levels are sufficiently low, i.e. much lower than the lowest now attained. A search for this phenomenon may be worthwhile.

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